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Real embedding and equivariant eta forms

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Abstract
Bismut and Zhang (Math Ann 295(4):661–684, 1993) establish a mod $\mathbb{Z}$ embedding formula of Atiyah–Patodi–Singer reduced eta invariants. In this paper, we explain the hidden mod $\mathbb{Z}$ term as a spectral flow and extend this embedding formula to the equivariant family case. In this case, the spectral flow is generalized to the equivariant Chern character of some equivariant Dai–Zhang higher spectral flow.

Keywords Equivariant eta form · Index theory and fixed point theory · Higher spectral flow · Direct image

Mathematics Subject Classification 58J20 · 58J28 · 58J30 · 58J35

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1 Introduction

Let $i\colon Y \to X$ be an embedding between two odd dimensional compact oriented spin manifolds. For any Hermitian vector bundle μ over Y carrying a Hermitian connection, under a natural assumption, Bismut and Zhang [13] establish a mod $\mathbb{Z}$ formula, expressing the Atiyah–Patodi–Singer reduced eta invariant [2] of certain direct image of μ over X, through the reduced eta invariant of the bundle μ over Y, up to some geometric Chern–Simons current.

In this paper, we explain the hidden mod $\mathbb{Z}$ term as a spectral flow in Bismut–Zhang embedding formula and extend this embedding formula to the equivariant family case. In this case, the spectral flow is generalized to the equivariant Chern character of some equivariant Dai–Zhang higher spectral flow [18].

The main motivation of this generalization is to look for a general Grothendieck–Riemann–Roch theorem in the equivariant differential K-theory, which is already established in many important cases [15,16,22]. Roughly speaking, the differential K-theory is the smooth version of the arithmetic K-theory in Arakelov geometry. Our main result here is expected to play the same role in the equivariant differential K-theory of the Bunke–Schick model [15,16,24] as the Bismut–Lebeau embedding formula [11] does in the proof of Arithmetic Grothendieck–Riemann–Roch theorem in Arakelov geometry.

In this paper, we do not assume the manifold is spin or spin^c. We consider the general Clifford modules.

Let $\pi\colon W \to B$ be a smooth submersion of smooth oriented manifolds with compact fibres Y. Let $TY = TW/B$ be the relative tangent bundle to the fibres Y. Let $T^H_\pi W$ be a horizontal subbundle of TW. Let g^{TY} be a Riemannian metric on TY. Let $C(TY)$ be the Clifford algebra bundle of (TY, g^{TY}) and $(\mathcal{E}, h^{\mathcal{E}})$ be a $\mathbb{Z}_2$-graded self-adjoint $C(TY)$-module with Clifford connection $\nabla^{\mathcal{E}}$ (cf. (2.16) and (2.17)). Let G be a compact Lie group which acts fiberwisely on W, i.e., for any $g \in G$, $\pi \circ g = \pi$. We assume that the action of G preserves the horizontal bundle $T^H_\pi W$ and the orientation of TY and could be lifted on $\mathcal{E}$ such that it is compatible with the Clifford action and the $\mathbb{Z}_2$-grading. We assume that g^{TY}, $h^{\mathcal{E}}$, $\nabla^{\mathcal{E}}$ are G-invariant. For any $g \in G$, the equivariant Bismut–Cheeger eta form $\tilde{\eta}_g(\mathcal{F}, \mathcal{A}) \in \Omega^*(B, \mathbb{C})/\mathrm{Im}\, d$ is defined in Definition 2.4 up to exact forms with respect to the equivariant geometric family $\mathcal{F} = (W, \mathcal{E}, T^H_\pi W, g^{TY}, h^{\mathcal{E}}, \nabla^{\mathcal{E}})$ (cf. Definition 2.1) over B and a perturbation operator $\mathcal{A}$ (cf. Definition 2.3). Remark that if B is a point and $\dim Y$ is odd, then the equivariant eta form here degenerates to the canonical equivariant reduced eta invariant by taking a special perturbation operator [24, Remark 2.20].

Let $i\colon W \to V$ be an equivariant embedding of smooth G-equivariant oriented manifolds with even codimension. Let $\pi_V\colon V \to B$ be a G-equivariant submersion with compact fibres X, whose restriction $\pi_W\colon W \to B$ is an equivariant submersion with compact fibres Y. We assume that G acts on B trivially and the normal bundle $N_{Y/X}$ to Y in X has an equivariant Spin^c structure.

$$\begin{array}{ccccc} Y & \longrightarrow & W & & \\ {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & \overset{\pi_W}{\searrow} & \\ X & \longrightarrow & V & \xrightarrow{\pi_V} & B. \end{array}$$

Let $\mathcal{F}_Y = (W, \mathcal{E}_Y, T^H_{\pi_W} W, g^{TY}, h^{\mathcal{E}_Y}, \nabla^{\mathcal{E}_Y})$ and $\mathcal{F}_X = (V, \mathcal{E}_X, T^H_{\pi_V} V, g^{TX}, h^{\mathcal{E}_X}, \nabla^{\mathcal{E}_X})$ be two equivariant geometric families over B such that $(T^H_{\pi_W} W, g^{TY})$ and $(T^H_{\pi_V} V, g^{TX})$ satisfy the totally geodesic condition (3.11).

We state our main result of this paper as follows.

Theorem 1.1 *Assume that $N_{Y/X}$ has an equivariant Spin^c structure and the equivariant geometric families $\mathcal{F}_Y$ and $\mathcal{F}_X$ satisfy the fundamental assumption* (3.13) *and* (3.15)*. Let $\mathcal{A}_Y$ and $\mathcal{A}_X$ be the perturbation operators with respect to $\mathcal{F}_Y$ and $\mathcal{F}_X$. Then for any compact submanifold K of B, there exists $T_0 > 2$, which depends on K, such that for any $T \geq T_0$, modulo exact forms on B, over K, we have*

$$\tilde{\eta}_g(\mathcal{F}_X, \mathcal{A}_X) = \tilde{\eta}_g(\mathcal{F}_Y, \mathcal{A}_Y) + \int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX})\, \gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X) + \mathrm{ch}_g(\mathrm{sf}_G\{D(\mathcal{F}_X) + \mathcal{A}_X, D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y}\}). \tag{1.1}$$

Here $\gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X)$ is the equivariant Bismut–Zhang current defined in Definition 3.3 *and $\mathcal{A}_{T,Y}$ is the operator given in Proposition* 3.6*. The last term in* (1.1) *is the equivariant Chern character of the equivariant Dai–Zhang higher spectral flow which explains the* mod $\mathbb{Z}$ *term in the original Bismut–Zhang embedding formula.*

The proof of our main result here is highly related to the analytical localization technique developed in [7,8,11,12]. Thanks to the functoriality of equivariant eta forms proved in [24,25], we only need to prove the embedding formula when B is a point and dim X is odd.

Note that in [21,31], the authors give another proof of the Bismut–Zhang embedding formula without using the analytical localization technique. It is interesting to ask whether there is another proof of our main result here from that line.

Our paper is organized as follows. In Sect. 2, we summarize the definition and the properties of equivariant eta forms in [24] using the language of Clifford modules. In Sect. 3, we state our main result. We also discuss an application on the equivariant Atiyah–Hirzebruch direct image. In Sect. 4, we prove our main result in two steps. In Sect. 4.1, we prove Theorem 1.1 when the base space B is a point following [13]. In Sect. 4.2, we explain how to use the functoriality to reduce the proof of Theorem 1.1 to the case considered in Sect. 4.1.

Notation We use the Einstein summation convention in this paper.

We also use the superconnection formalism of Quillen [28] and Bismut–Cheeger [9]. If A is a $\mathbb{Z}_2$-graded algebra, and if $a, b \in A$, then we will note $[a, b]$ as the supercommutator of a, b. If B is another $\mathbb{Z}_2$-graded algebra, we will note $A \widehat{\otimes} B$ as the $\mathbb{Z}_2$-graded tensor product.

For a fibre bundle $\pi : V \to B$, we will often use the integration of the differential forms along the oriented fibres X in this paper. Since the fibres may be odd dimensional, we must make precise our sign conventions: for $\alpha \in \Omega^*(B)$ and $\beta \in \Omega^*(V)$, then

$$\int_X (\pi^*\alpha) \wedge \beta = \alpha \wedge \int_X \beta. \tag{1.2}$$

2 Equivariant eta forms

In this section, we summarize the definition and the properties of equivariant eta forms in [24,25] using the language of Clifford modules. Note that locally all manifolds are spin. The proofs of them are the same as in the spin case. In Sect. 2.1, we recall elementary results on Clifford algebras. In Sect. 2.2, we describe the geometry of the fibration and recall the Bismut superconnection. In Sect. 2.3, we define the equivariant eta form and state the anomaly formula. In Sect. 2.4, we explain the functoriality of the equivariant eta forms.

2.1 Clifford algebras

Let E be an oriented Euclidean vector space, such that $\dim E = n$, with orthonormal basis $\{e_i\}_{1\leq i\leq n}$. Let $C(E)$ be the complex Clifford algebra of E defined by the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}. \tag{2.1}$$

Sometimes, we also denote by $c(e)$ the element of $C(E)$ corresponding to $e \in E$. Let Spin_n^c be the Spin^c group associated with $C(E)$ (see [23, Appendix D]).

If $e \in E$, let $e^* \in E^*$ correspond to e by the scalar product of E. The exterior algebra $\Lambda(E^*) \otimes_{\mathbb{R}} \mathbb{C}$ is a module of $C(E)$ defined by

$$c(e)\alpha = e^* \wedge \alpha - \iota_e \alpha \tag{2.2}$$

for any $\alpha \in \Lambda(E^*) \otimes_{\mathbb{R}} \mathbb{C}$. The map $a \mapsto c(a)\cdot 1$, $a \in C(E)$, induces an isomorphism of vector spaces

$$\sigma : C(E) \to \Lambda(E^*) \otimes_{\mathbb{R}} \mathbb{C}. \tag{2.3}$$

If n is even, up to isomorphism, $C(E)$ has a unique irreducible module, the spinor $\mathcal{S}(E)$, which is $\mathbb{Z}_2$-graded. We denote by $\mathcal{S}(E) = \mathcal{S}_+(E) \oplus \mathcal{S}_-(E)$. Moreover, there are isomorphisms of $\mathbb{Z}_2$-graded algebras

$$C(E) \simeq \mathrm{End}(\mathcal{S}(E)) \simeq \mathcal{S}(E)\widehat{\otimes}\mathcal{S}(E)^*. \tag{2.4}$$

Note that $\mathcal{S}(E)$ is a representation of Spin_n^c induced by the Clifford action.

If n is odd, $C(E)$ has two (inequivalent) irreducible modules. However, their restriction to Spin_n^c are equivalent irreducible representations, which we denote by $\mathcal{S}(E)$. We have

$$C(E) \simeq \mathrm{End}(\mathcal{S}(E)) \oplus \mathrm{End}(\mathcal{S}(E)). \tag{2.5}$$

Let F be another oriented Euclidean vector space. Then

$$C(E \oplus F) \simeq C(E)\widehat{\otimes}C(F). \tag{2.6}$$

Let us introduce the convention for the tensor product of Clifford modules which will be used in the whole paper. Let $\mathcal{X}$ and $\mathcal{Y}$ be two representations of $C(E)$ and $C(F)$. If both $\dim E$ and $\dim F$ are odd, put

$$\mathcal{X}\widehat{\otimes}\mathcal{Y} = \mathcal{X} \otimes \mathcal{Y} \otimes \mathbb{C}^2. \tag{2.7}$$

By [9, (1.10)], $\mathcal{X}\widehat{\otimes}\mathcal{Y}$ is a $\mathbb{Z}_2$-representation of $C(E)\widehat{\otimes}C(F)$. If $\dim E$ is odd, if $\dim F$ is even, and if $\mathcal{Y}$ is $\mathbb{Z}_2$-graded, then put

$$\mathcal{X}\widehat{\otimes}\mathcal{Y} = \mathcal{X} \otimes \mathcal{Y}. \tag{2.8}$$

By [9, (1.11)], $\mathcal{X}\widehat{\otimes}\mathcal{Y}$ is a representation of $C(E)\widehat{\otimes}C(F)$. If both $\dim E$ and $\dim F$ are even and if both $\mathcal{X}$, $\mathcal{Y}$ are $\mathbb{Z}_2$-graded, we define $\mathcal{X}\widehat{\otimes}\mathcal{Y}$ as (2.8) with the same action defined in [9, (1.11)]. Then $\mathcal{X}\widehat{\otimes}\mathcal{Y}$ is a $\mathbb{Z}_2$-representation of $C(E)\widehat{\otimes}C(F)$.

With the above convention, we have the isomorphism of Clifford modules

$$\mathcal{S}(E \otimes F) \simeq \mathcal{S}(E)\widehat{\otimes}\mathcal{S}(F). \tag{2.9}$$

2.2 Bismut superconnection

Let $\pi : W \to B$ be a smooth submersion of smooth oriented manifolds with compact fibres Y. We assume that B is connected. Remark that W here is not assumed to be connected. To simplify the notations, we usually denote the connected component by W when there is no confusion.

Let $TY = TW/B$ be the relative tangent bundle to the fibres Y. Then TY is orientable. Let $T^H_\pi W$ be a horizontal subbundle of TW such that

$$TW = T^H_\pi W \oplus TY. \tag{2.10}$$

The splitting (2.10) gives an identification

$$T^H_\pi W \cong \pi^* TB. \tag{2.11}$$

If there is no ambiguity, we will omit the subscript π in $T^H_\pi W$.

Let g^{TY}, g^{TB} be Riemannian metrics on TY, TB. We equip $TW = T^H W \oplus TY$ with the Riemannian metric

$$g^{TW} = \pi^* g^{TB} \oplus g^{TY}. \tag{2.12}$$

Let ∇^{TW}, ∇^{TB} be the Levi-Civita connections of (W, g^{TW}), (B, g^{TB}). Let P^{TY} be the projection $P^{TY} : TW = T^H W \oplus TY \to TY$. Set

$$\nabla^{TY} = P^{TY} \nabla^{TW} P^{TY}. \tag{2.13}$$

Then ∇^{TY} is a Euclidean connection on TY.

Let $\nabla^{TB,TY}$ be the connection on $TW = T^H W \oplus TY$ defined by

$$\nabla^{TB,TY} = \pi^* \nabla^{TB} \oplus \nabla^{TY}. \tag{2.14}$$

Then $\nabla^{TB,TY}$ preserves the metric g^{TW} in (2.12).

Set

$$S = \nabla^{TW} - \nabla^{TB,TY}. \tag{2.15}$$

Then S is a 1-form on W with values in antisymmetric elements of $\mathrm{End}(TW)$. By [6, Theorem 1.9], we know that ∇^{TY} and the $(3,0)$-tensor $g^{TW}(S(\cdot)\cdot, \cdot)$ only depend on $(T^H W, g^{TY})$.

Let $C(TY)$ be the Clifford algebra bundle of (TY, g^{TY}), whose fibre at $x \in W$ is the Clifford algebra $C(T_x Y)$ of the Euclidean vector space $(T_x Y, g^{T_x Y})$. A $\mathbb{Z}_2$-graded self-adjoint $C(TY)$-module,

$$\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-, \tag{2.16}$$

is a $\mathbb{Z}_2$-graded vector bundle equipped with a Hermitian metric $h^{\mathcal{E}}$ preserving the splitting (2.16) and a fiberwise Clifford multiplication c of $C(TY)$ such that the action c restricted to TY is skew-adjoint on $(\mathcal{E}, h^{\mathcal{E}})$ and anticommutes (resp. commutes) with the $\mathbb{Z}_2$-grading if the dimension of the fibres is even (resp. odd). Let $\tau^{\mathcal{E}}$ be the $\mathbb{Z}_2$-grading of $\mathcal{E}$ which is ± 1 on $\mathcal{E}_\pm$.

Let $\nabla^{\mathcal{E}}$ be a Clifford connection on $\mathcal{E}$ associated with ∇^{TY}, that is, $\nabla^{\mathcal{E}}$ preserves $h^{\mathcal{E}}$ and the splitting (2.16) and for any $U \in TW$, $Z \in \mathscr{C}^\infty(W, TY)$,

$$[\nabla^{\mathcal{E}}_U, c(Z)] = c(\nabla^{TY}_U Z). \tag{2.17}$$

Let $\{W_\alpha\}$ be the connected components of W. Then $\pi|_{W_\alpha} : W_\alpha \to B$ is a smooth submersion with compact fibres Y_α.

For a locally oriented orthonormal basis $e_1, \dots, e_{\dim Y_\alpha}$ of TY_α, we define the chirality operator on $\mathcal{E}|_{W_\alpha}$ by

$$\Gamma_\alpha = \begin{cases} (\sqrt{-1})^{\dim Y_\alpha/2} c(e_1) \cdots c(e_{\dim Y_\alpha}), & \text{if } \dim Y_\alpha \text{ is even;} \\ \mathrm{Id}_{\mathcal{E}|_{W_\alpha}}, & \text{if } \dim Y_\alpha \text{ is odd.} \end{cases} \tag{2.18}$$

Note that our definition here is different from [5, Lemma 3.17] when $\dim Y_\alpha$ is odd. Then Γ_α does not depend on the choice of the basis and is globally defined. We note that $\Gamma_\alpha^2 = 1$ and $[\tau^{\mathcal{E}}, \Gamma_\alpha] = 0$. Set

$$\tau_\alpha^{\mathcal{E}/\mathcal{S}} = \tau^{\mathcal{E}} \cdot \Gamma_\alpha. \tag{2.19}$$

Then $(\tau_\alpha^{\mathcal{E}/\mathcal{S}})^2 = 1$.

Locally, we could write

$$\mathcal{E}|_{W_\alpha} = \mathcal{S}_0(TY_\alpha) \widehat{\otimes} \xi, \tag{2.20}$$

where $\mathcal{S}_0(TY_\alpha)$ is the spinor bundle for the (possibly non-existent) spin structure of TY_α and $\xi = \xi_+ \oplus \xi_-$ is a $\mathbb{Z}_2$-graded vector bundle. Then Γ_α, $\tau_\alpha^{\mathcal{E}/\mathcal{S}}$ and $\tau^{\mathcal{E}}$ correspond to the $\mathbb{Z}_2$-gradings of $\mathcal{S}_0(TY_\alpha)$, ξ and $\mathcal{S}_0(TY_\alpha) \widehat{\otimes} \xi$.

Let G be a compact Lie group which acts on W and B such that for any $g \in G, \pi \circ g = g \circ \pi$. We assume that the action of G preserves the splitting (2.10) and the orientation of TY and could be lifted on $\mathcal{E}$ such that it is compatible with the Clifford action and preserves the splitting (2.16). We assume that g^{TY}, $h^{\mathcal{E}}$, $\nabla^{\mathcal{E}}$ are G-invariant.

Definition 2.1 (Compare with [15, Definition 2.2], [24, Definition 1.1]) An equivariant geometric family $\mathcal{F}$ over B is a family of G-equivariant geometric data

$$\mathcal{F} = (W, \mathcal{E}, T^H W, g^{TY}, h^{\mathcal{E}}, \nabla^{\mathcal{E}}) \tag{2.21}$$

described as above. We call the equivariant geometric family $\mathcal{F}$ is even (resp. odd) if for any connected component of fibres, the dimension of it is even (resp. odd).

Let $D(\mathcal{F})$ be the fiberwise Dirac operator

$$D(\mathcal{F}) = c(e_i) \nabla^{\mathcal{E}}_{e_i} \tag{2.22}$$

associated with the equivariant geometric family $\mathcal{F}$. Then the G-action commutes with $D(\mathcal{F})$.

For $b \in B$, let $\mathscr{E}_b$ be the set of smooth sections over Y_b of $\mathcal{E}_b$. As in [6], we will regard $\mathscr{E}$ as an infinite dimensional fibre bundle over B. If $V \in TB$, let $V^H \in T^H W$ be its horizontal lift in $T^H W$ so that $\pi_* V^H = V$. For any $V \in TB$, $s \in \mathscr{C}^\infty(B, \mathscr{E}) = \mathscr{C}^\infty(W, \mathcal{E})$, the connection

$$\nabla_V^{\mathscr{E},u} s := \nabla^{\mathcal{E}}_{V^H} s - \frac{1}{2} \langle S(e_i) e_i, V^H \rangle\, s \tag{2.23}$$

preserves the L^2-product on $\mathscr{E}$ (see e.g., [10, Proposition 1.4]). Let $\{f_p\}$ be a local frame of TB and $\{f^p\}$ be its dual. We denote by $\nabla^{\mathscr{E},u} = f^p \wedge \nabla^{\mathscr{E},u}_{f_p}$. We denote by $c(T^H) = -\frac{1}{2} c(P^{TY}[f_p^H, f_q^H]) f^p \wedge f^q \wedge$. By [6, (3.18)], the rescaled Bismut superconnection $\mathbb{B}_u : \mathscr{C}^\infty(B, \Lambda(T^*B) \widehat{\otimes} \mathscr{E}) \to \mathscr{C}^\infty(B, \Lambda(T^*B) \widehat{\otimes} \mathscr{E})$ is defined by

$$\mathbb{B}_u = \sqrt{u} D(\mathcal{F}) + \nabla^{\mathscr{E},u} - \frac{1}{4\sqrt{u}} c(T^H). \tag{2.24}$$

Obviously, the Bismut superconnection $\mathbb{B}_u$ commutes with the G-action. Furthermore, $\mathbb{B}_u^2$ is a 2-order elliptic differential operator along the fibres Y. Let $\exp(-\mathbb{B}_u^2)$ be the family of heat operators associated with the fiberwise elliptic operator $\mathbb{B}_u^2$. Then $\exp(-\mathbb{B}_u^2)$ is a smooth family of smoothing operators (see e.g., [5, Theorem 9.51]).

Let P be a section of $\Lambda(T^*B)\widehat{\otimes}\,\mathrm{End}(\mathscr{E})$. Set

$$\mathrm{Tr}_s[P] := \mathrm{Tr}[\tau^{\mathcal{E}} P] \in \Lambda(T^*B). \tag{2.25}$$

Here the trace operator on the right hand side of (2.25) only acts on $\mathcal{E}$. We use the convention that if $\omega \in \Lambda(T^*B)$,

$$\mathrm{Tr}_s[\omega P] = \omega\,\mathrm{Tr}_s[P]. \tag{2.26}$$

It is compatible with the sign convention (1.2). We denote by $\mathrm{Tr}_s^{\mathrm{odd/even}}[P]$ the part of $\mathrm{Tr}_s[P]$ which takes values in odd or even forms. Set

$$\widetilde{\mathrm{Tr}}[P] = \begin{cases} \mathrm{Tr}_s[P], & \text{if } \dim Y \text{ is even;} \\ \mathrm{Tr}_s^{\mathrm{odd}}[P], & \text{if } \dim Y \text{ is odd.} \end{cases} \tag{2.27}$$

2.3 Equivariant eta forms

In this subsection, we state the definition and the anomaly formula of equivariant eta forms in the language of Clifford modules.

In the rest of the paper, we assume that G acts trivially on B.

Take $g \in G$ and set $W^g = \{x \in W : gx = x\}$. Then W^g is a submanifold of W and $\pi|_{W^g} : W^g \to B$ is a fibre bundle with compact fibres Y^g. Let $N_{W^g/W}$ denote the normal bundle of W^g in W, then $N_{W^g/W} = TW/TW^g = TY/TY^g$. We also denote it by $N_{Y^g/Y}$.

The differential of g gives a bundle isometry $dg : N_{Y^g/Y} \to N_{Y^g/Y}$. Since g lies in a compact abelian Lie group, we know that there is an orthonormal decomposition of smooth vector bundles over W^g

$$TY|_{W^g} = TY^g \oplus N_{Y^g/Y} = TY^g \oplus \bigoplus_{0<\theta\leq\pi} N(\theta), \tag{2.28}$$

where $dg|_{N(\pi)} = -\mathrm{Id}$ and for each $\theta, 0 < \theta < \pi$, $N(\theta)$ is a complex vector bundle on which dg acts by multiplication by $e^{i\theta}$. Since g preserves the metric and the orientation of TY, $\det(dg|_{N(\pi)}) = 1$. Thus $\dim N(\pi)$ is even. So the normal bundle $N_{Y^g/Y}$ is even dimensional.

Observe that if $N(\pi) = 0$ or if TY has a G-equivariant Spin^c structure, then TY^g is canonically oriented (cf. [5, Proposition 6.14], [25, Proposition 2.1]). In general, TY^g is not necessary orientable. In this paper we assume that TY^g is orientable and fix an orientation of TY^g. In this case $N_{Y^g/Y}$ is canonically oriented.

Let E be an equivariant real Euclidean vector bundle over W. We could get the decomposition of real vector bundles over W^g in the same way as (2.28),

$$E|_{W^g} = \bigoplus_{0\leq\theta\leq\pi} E(\theta). \tag{2.29}$$

Here we also denote $E(0)$ by E^g.

Let ∇^E be an equivariant Euclidean connection on E. Then it preserves the decomposition (2.29). Let ∇^{E^g} and $\nabla^{E(\theta)}$ be the corresponding induced connections on E^g and $E(\theta)$, and let R^{E^g} and $R^{E(\theta)}$ be the corresponding curvatures.

Set

$$\widehat{\mathrm{A}}_g(E, \nabla^E) = \det{}^{\frac{1}{2}}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{E^g}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{E^g}\right)}\right) \cdot \prod_{0<\theta\le\pi}\left(\sqrt{-1}^{\frac{1}{2}\dim_{\mathbb{R}} E(\theta)}\det{}^{\frac{1}{2}}\left(1 - g\exp\left(\frac{\sqrt{-1}}{2\pi}R^{E(\theta)}\right)\right)\right)^{-1}. \quad (2.30)$$

Let $\mathrm{End}_{C(TY)}(\mathcal{E})$ be the set of endomorphisms of $\mathcal{E}$ supercommuting with the Clifford action. Then it is a vector bundle over W. Let $\mathrm{End}_{C(TY)}(\mathcal{E})_x$ be the fiber at $x \in W$. For any $a \in \mathrm{End}_{C(TY)}(\mathcal{E})_x$ and $x \in W_\alpha$, we define the relative trace $\mathrm{Tr}^{\mathcal{E}/\mathcal{S}}\colon \mathrm{End}_{C(TY)}(\mathcal{E})_x \to \mathbb{C}$ by (cf. [5, Definition 3.28])

$$\mathrm{Tr}^{\mathcal{E}/\mathcal{S}}[a] = \begin{cases} 2^{-\dim Y_\alpha/2}\,\mathrm{Tr}_s[\Gamma_\alpha a], & \text{if } \dim Y_\alpha \text{ is even;} \\ 2^{-(\dim Y_\alpha - 1)/2}\,\mathrm{Tr}_s[a], & \text{if } \dim Y_\alpha \text{ is odd.} \end{cases} \quad (2.31)$$

The relative trace could be naturally extended on $\mathscr{C}^\infty(W, \pi^*\Lambda(T^*B) \otimes \mathrm{End}_{C(TY)}(\mathcal{E}))$ as in (2.26).

Let $R^{\mathcal{E}}$ be the curvature of $\nabla^{\mathcal{E}}$. Let

$$R^{\mathcal{E}/\mathcal{S}} := R^{\mathcal{E}} - \frac{1}{4}\langle R^{TY}e_i, e_j\rangle c(e_i)c(e_j) \in \mathscr{C}^\infty(W, \pi^*\Lambda(T^*B) \otimes \mathrm{End}_{C(TY)}(\mathcal{E})) \quad (2.32)$$

be the twisting curvature of the $C(TY)$-module $\mathcal{E}$ as in [5, Proposition 3.43].

By [5, Lemma 6.10], along W^g, the action of $g \in G$ on $\mathcal{E}$ may be identified with a section $g^{\mathcal{E}}$ of $C(N_{Y^g/Y}) \otimes_{\mathbb{C}} \mathrm{End}_{C(TY)}(\mathcal{E})$. Let $\dim N_{Y^g/Y} = \ell_1$. Under the isomorphism (2.3), $\sigma(g^{\mathcal{E}}) \in \mathscr{C}^\infty(W^g, \Lambda N^*_{Y^g/Y} \otimes_{\mathbb{R}} \mathrm{End}_{C(TY)}(\mathcal{E}))$. Since we assume that $N_{Y^g/Y}$ is oriented, paring with the volume form, we could get the highest degree coefficient $\sigma_{\ell_1}(g^{\mathcal{E}}) \in \mathscr{C}^\infty(W^g, \mathrm{End}_{C(TY)}(\mathcal{E}))$ of $\sigma(g^{\mathcal{E}})$.

Then we could define the localized relative Chern character $\mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) \in \Omega^*(W^g, \mathbb{C})$ in the same way as [5, Definition 6.13] by

$$\mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) := \frac{2^{\ell_1/2}}{\det^{1/2}(1 - g|_{N_{Y^g/Y}})}\mathrm{Tr}^{\mathcal{E}/\mathcal{S}}\left[\sigma_{\ell_1}(g^{\mathcal{E}})\exp\left(-\frac{R^{\mathcal{E}/\mathcal{S}}|_{W^g}}{2\pi\sqrt{-1}}\right)\right]. \quad (2.33)$$

Note that if TY has an equivariant spin structure, the localized relative Chern character here is just the usual equivariant Chern character.

Recall that if B is compact, the equivariant K-group $K^0_G(B)$ is the Grothendieck group of the equivalent classes of the equivariant vector bundles over B. Let $\iota\colon B \to B \times S^1$ be a G-equivariant inclusion map. It is well known that if the G-action on S^1 is trivial,

$$K^1_G(B) \simeq \ker\left(\iota^* : K^0_G(B \times S^1) \to K^0_G(B)\right). \quad (2.34)$$

For $x \in K^0_G(B)$, $g \in G$, the classical equivariant Chern character map sends x to $\mathrm{ch}_g(x) \in H^{\mathrm{even}}(B, \mathbb{C})$. By (2.34), for $x \in K^1_G(B)$, we can regard x as an element x' in $K^0_G(B \times S^1)$. The odd equivariant Chern character map

$$\mathrm{ch}_g\colon K^1_G(B) \longrightarrow H^{\mathrm{odd}}(B, \mathbb{C}) \quad (2.35)$$

is defined by (cf. e.g., [25, (2.52)])

$$\mathrm{ch}_g(x) := \int_{S^1}\mathrm{ch}_g(x'). \quad (2.36)$$

We adopt the sign convention as in (1.2).

Furthermore, the classical construction of Atiyah–Singer [3,4] assigns to each even (resp. odd) equivariant geometric family $\mathcal{F}$ its equivariant (analytic) index $\mathrm{ind}(D(\mathcal{F})) \in K_G^0(B)$ (resp. $K_G^1(B)$).

For $\alpha \in \Omega^j(B)$, set

$$\psi_B(\alpha) = \begin{cases} \left(2\pi\sqrt{-1}\right)^{-\frac{j}{2}} \cdot \alpha, & \text{if } j \text{ is even;} \\ \frac{1}{\sqrt{\pi}}\left(2\pi\sqrt{-1}\right)^{-\frac{j-1}{2}} \cdot \alpha, & \text{if } j \text{ is odd.} \end{cases} \tag{2.37}$$

The following family local index theorem is a well-known result (see e.g., [25, Theorem 2.2]).

Theorem 2.2 *For any $u > 0$ and $g \in G$, the differential form $\psi_B \widetilde{\mathrm{Tr}}[g\exp(-\mathbb{B}_u^2)] \in \Omega^*(B, \mathbb{C})$ is closed and its cohomology class is independent of $u > 0$. As $u \to 0$,*

$$\lim_{u\to 0} \psi_B \widetilde{\mathrm{Tr}}[g\exp(-\mathbb{B}_u^2)] = \int_{Y^g} \widehat{\mathrm{A}}_g(TY, \nabla^{TY})\, \mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}). \tag{2.38}$$

If B is compact, the closed form $\psi_B \widetilde{\mathrm{Tr}}[g\exp(-\mathbb{B}_u^2)]$ represents the class $\mathrm{ch}_g(\mathrm{ind}(D(\mathcal{F})))$.

Definition 2.3 [24, Definition 2.10] A perturbation operator with respect to $D(\mathcal{F})$, denoted by $\mathcal{A}$, is defined to be a smooth family of G-equivariant bounded self-adjoint pseudodifferential operators on $\mathcal{E}$ along the fibres such that it commutes (resp. anti-commutes) with the $\mathbb{Z}_2$-grading of $\mathcal{E}$ when the fibres are odd (resp. even) dimensional, and $D(\mathcal{F}) + \mathcal{A}$ is invertible.

Remark that from [24, Proposition 2.3], if B is compact and at least one component of the fibres has the non-zero dimension, then there exists a perturbation operator with respect to $D(\mathcal{F})$ if and only if $\mathrm{ind}(D(\mathcal{F})) = 0 \in K_G^*(B)$.

In the followings, we always assume that there exists a perturbation operator with respect to $D(\mathcal{F})$ on $\mathcal{F}$.

For $\alpha \in \Lambda(T^*(\mathbb{R} \times B))$, we can expand α in the form

$$\alpha = du \wedge \alpha_0 + \alpha_1, \quad \alpha_0, \alpha_1 \in \Lambda(T^*B). \tag{2.39}$$

Set

$$[\alpha]^{du} := \alpha_0. \tag{2.40}$$

Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$ be a cut-off function such that

$$\chi(u) = \begin{cases} 0, & \text{if } u \leq 1; \\ 1, & \text{if } u \geq 2. \end{cases} \tag{2.41}$$

Let $\mathcal{A}$ be a perturbation operator with respect to $D(\mathcal{F})$. Then $\mathcal{A}$ could be extended to $1\widehat{\otimes}\mathcal{A}$ on $\mathscr{C}^\infty(B, \pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E})$ as an element of the $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded algebras. In this case, we have

$$(\alpha\widehat{\otimes}1)(1\widehat{\otimes}\mathcal{A}) = (-1)^{\deg\alpha}(1\widehat{\otimes}\mathcal{A})(\alpha\widehat{\otimes}1). \tag{2.42}$$

We usually abbreviate $1\widehat{\otimes}\mathcal{A}$ by $\mathcal{A}$ when there is no confusion. Set

$$\mathbb{B}'_u = \mathbb{B}_u + \sqrt{u}\chi(\sqrt{u})\mathcal{A}. \tag{2.43}$$

Definition 2.4 [24, Definition 2.11] For any $g \in G$, modulo exact forms on B, the equivariant Bismut–Cheeger eta form with perturbation operator $\mathcal{A}$ is defined by

$$\tilde{\eta}_g(\mathcal{F}, \mathcal{A}) := -\int_0^\infty \left\{ \psi_{\mathbb{R}\times B}\, \widetilde{\mathrm{Tr}} \left[g \exp\left(-\left(\mathbb{B}'_u + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right] \right\}^{du} du \\ \in \Omega^*(B, \mathbb{C})/\mathrm{Im}\, d. \tag{2.44}$$

Remark that by our convention in Sect. 2.1, du anti-commutes with $\mathcal{A}$ and $c(v)$ for any $v \in TY$.

From the discussion in [24, Section 2.3], the equivariant eta form with perturbation in Definition 2.4 is well defined and does not depend on the cut-off function. Moreover, since we assume that Y^g is oriented, we have (cf. [24, (2.44)])

$$d^B \tilde{\eta}_g(\mathcal{F}, \mathcal{A}) = \int_{Y^g} \widehat{\mathrm{A}}_g(TY, \nabla^{TY})\, \mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}). \tag{2.45}$$

Remark 2.5 After changing the variable, we have

$$\tilde{\eta}_g(\mathcal{F}, \mathcal{A}) = -\int_0^\infty \left\{ \psi_{\mathbb{R}\times B}\, \widetilde{\mathrm{Tr}} \left[g \exp\left(-\left(\mathbb{B}'_{u^2} + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right] \right\}^{du} du. \tag{2.46}$$

We will often use this formula as the definition of the equivariant eta form.

Explicitly,

$$\tilde{\eta}_g(\mathcal{F}, \mathcal{A}) = \begin{cases} \displaystyle\int_0^\infty \frac{1}{\sqrt{\pi}} \psi_B\, \mathrm{Tr}_s^{\mathrm{even}} \left[g\, \frac{\partial \mathbb{B}'_{u^2}}{\partial u} \exp(-(\mathbb{B}'_{u^2})^2) \right] du \\ \qquad\qquad \in \Omega^{\mathrm{even}}(B, \mathbb{C})/\mathrm{Im}\, d, \text{ if } \mathcal{F} \text{ is odd;} \\ \displaystyle\int_0^\infty \frac{1}{2\sqrt{\pi}\sqrt{-1}} \psi_B\, \mathrm{Tr}_s \left[g\, \frac{\partial \mathbb{B}'_{u^2}}{\partial u} \exp(-(\mathbb{B}'_{u^2})^2) \right] du \\ \qquad\qquad \in \Omega^{\mathrm{odd}}(B, \mathbb{C})/\mathrm{Im}\, d, \text{ if } \mathcal{F} \text{ is even.} \end{cases} \tag{2.47}$$

From [24, Remark 2.20], when B is a point, $\dim Y$ is odd, letting $\mathcal{A} = P_{\ker D(\mathcal{F}_Y)}$ be the orthogonal projection onto the kernel of $D(\mathcal{F}_Y)$, the equivariant eta form $\tilde{\eta}_g(\mathcal{F}, \mathcal{A})$ is just the equivariant reduced eta invariant defined in [20]. Note that from (2.47), if B is a point and $\dim Y$ is even, we have $\tilde{\eta}_g(\mathcal{F}, \mathcal{A}) = 0$ for any perturbation operator $\mathcal{A}$.

Let $\mathcal{F} = (W, \mathcal{E}, T^H W, g^{TY}, h^{\mathcal{E}}, \nabla^{\mathcal{E}})$ and $\mathcal{F}' = (W, \mathcal{E}, T'^H W, g'^{TY}, h'^{\mathcal{E}}, \nabla'^{\mathcal{E}})$ be two equivariant geometric families over B. Let

$$\left(\widetilde{\widehat{\mathrm{A}}}_g \cdot \widetilde{\mathrm{ch}}_g\right)(\nabla^{TY}, \nabla'^{TY}, \nabla^{\mathcal{E}}, \nabla'^{\mathcal{E}}) \in \Omega^*(W^g, \mathbb{C})/\mathrm{Im} d$$

be the Chern–Simons class (cf. [27, Appendix B]) such that

$$d\left(\widetilde{\widehat{\mathrm{A}}}_g \cdot \widetilde{\mathrm{ch}}_g\right)(\nabla^{TY}, \nabla'^{TY}, \nabla^{\mathcal{E}}, \nabla'^{\mathcal{E}}) \\ = \widehat{\mathrm{A}}_g(TY, \nabla'^{TY})\, \mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla'^{\mathcal{E}}) - \widehat{\mathrm{A}}_g(TY, \nabla^{TY})\, \mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}). \tag{2.48}$$

When B is compact, let $\mathrm{sf}_G\{(D(\mathcal{F}')+\mathcal{A}', P'), (D(\mathcal{F})+\mathcal{A}, P)\} \in K_G^*(B)$, which we often simply denote by $\mathrm{sf}_G\{D(\mathcal{F}')+\mathcal{A}', D(\mathcal{F})+\mathcal{A}\}$, be the equivariant Dai–Zhang higher spectral flow defined in [24, Definition 2.5, 2.6], where P, P' are the orthonormal projections onto the eigenspaces of positive eigenvalues with respect to $D(\mathcal{F})+\mathcal{A}$, $D(\mathcal{F}')+\mathcal{A}'$ respectively. If B is a point and dim Y is odd, it is just the canonical equivariant spectral flow.

The following anomaly formula is proved in [24, Theorem 2.17] and [25, Theorem 2.7].

Theorem 2.6 *Let $\mathcal{A}$, $\mathcal{A}'$ be perturbation operators with respect to $D(\mathcal{F})$, $D(\mathcal{F}')$ respectively. For any $g \in G$, modulo exact forms on B, we have*

(a) *if B is compact, then*

$$\tilde{\eta}_g(\mathcal{F}', \mathcal{A}') - \tilde{\eta}_g(\mathcal{F}, \mathcal{A}) = \int_{Y^g} \left(\widetilde{\widehat{\mathrm{A}}}_g \cdot \widetilde{\mathrm{ch}}_g\right)\left(\nabla^{TY}, \nabla'^{TY}, \nabla^{\mathcal{E}}, \nabla'^{\mathcal{E}}\right) + \mathrm{ch}_g(\mathrm{sf}_G\{D(\mathcal{F}')+\mathcal{A}', D(\mathcal{F})+\mathcal{A}\}); \tag{2.49}$$

(b) *if B is noncompact and there exists a smooth path $(\mathcal{F}_s, \mathcal{A}_s)$, $s \in [0,1]$, connecting $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{F}', \mathcal{A}')$ such that for any $s \in [0,1]$, $\mathcal{A}_s$ is the perturbation operator of $D(\mathcal{F}_s)$, then*

$$\tilde{\eta}_g(\mathcal{F}', \mathcal{A}') - \tilde{\eta}_g(\mathcal{F}, \mathcal{A}) = \int_{Y^g} \left(\widetilde{\widehat{\mathrm{A}}}_g \cdot \widetilde{\mathrm{ch}}_g\right)\left(\nabla^{TY}, \nabla'^{TY}, \nabla^{\mathcal{E}}, \nabla'^{\mathcal{E}}\right). \tag{2.50}$$

2.4 Functoriality

Let $\pi_M : U \to W$ be a G-equivariant submersion of smooth manifolds with compact oriented fibres M. Let $(\mathcal{E}_M, h^{\mathcal{E}_M})$ be a $\mathbb{Z}_2$-graded self-adjoint equivariant $C(TM)$-module. Let

$$\mathcal{F}_M = (U, \mathcal{E}_M, T^H_{\pi_M}U, g^{TM}, h^{\mathcal{E}_M}, \nabla^{\mathcal{E}_M}) \tag{2.51}$$

be a G-equivariant geometric family over W. Then $\pi_Z := \pi \circ \pi_M : U \to B$ is a G-equivariant submersion with compact oriented fibres Z, whose orientation is induced by the orientations of Y and M. Then we have the diagram of submersions:

$$\begin{array}{ccccc} M & \longrightarrow & Z & \longrightarrow & U \\ & & \downarrow & & \downarrow \pi_M \quad \searrow^{\pi_Z} \\ & & Y & \longrightarrow & W \xrightarrow{\ \pi\ } B. \end{array}$$

Set $T^H_{\pi_M}Z := T^H_{\pi_M}U \cap TZ$. Then we have the splitting of smooth vector bundles over U,

$$TZ = T^H_{\pi_M}Z \oplus TM, \tag{2.52}$$

and

$$T^H_{\pi_M}Z \cong \pi_M^* TY. \tag{2.53}$$

Take the geometric data $(T^H_{\pi_Z}U, g^{TZ}_T)$ of π_Z such that $T^H_{\pi_Z}U \subset T^H_{\pi_M}U$,

$$g^{TZ}_T = \pi_M^* g^{TY} \oplus \frac{1}{T^2} g^{TM} \tag{2.54}$$

and $g^{TZ} = g^{TZ}_1$. We denote the Clifford algebra bundle with respect to g^{TZ}_T by $C_T(TZ)$ and the corresponding 1-form in (2.15) by S_T.

Let $\{e_i\}$, $\{f_p\}$ be local orthonormal frames of TM, TY with respect to g^{TM}, g^{TY} respectively. Now $\{Te_i\}$ is a local orthonormal frame of TM with respect to the rescaled metric

$T^{-2}g^{TM}$. Let f_p^H be the horizontal lift of f_p with respect to (2.52). Now we define a Clifford algebra homomorphism

$$G_T\colon (C_T(TZ), g_T^{TZ}) \to (C(TZ), g^{TZ}) \tag{2.55}$$

by $G_T(c_T(f_p^H)) = c(f_p^H)$ and $G_T(c_T(Te_i)) = c(e_i)$. Under this homomorphism,

$$\mathcal{E}_Z := \pi_M^* \mathcal{E}_Y \widehat{\otimes} \mathcal{E}_M \tag{2.56}$$

with induced Hermitian metric $h^{\mathcal{E}_Z}$ is a $\mathbb{Z}_2$-graded self-adjoint equivariant $C_T(TZ)$-module.

Let

$${}^0\nabla^{\mathcal{E}_Z} := \pi_M^* \nabla^{\mathcal{E}_Y} \otimes 1 + 1 \otimes \nabla^{\mathcal{E}_M}. \tag{2.57}$$

Then it is a Clifford connection on $\mathcal{E}_Z$ associated with

$$\nabla^{TY,TM} := \pi_M^* \nabla^{TY} \otimes 1 + 1 \otimes \nabla^{TM}. \tag{2.58}$$

Now, we denote the Levi-Civita connection on TZ with respect to g_T^{TZ} by ∇_T^{TZ}. Then we could calculate that

$$\begin{aligned}\nabla_T^{\mathcal{E}_Z} := {}^0\nabla^{\mathcal{E}_Z} &+ \frac{1}{2}\langle S_T Te_i, f_p^H\rangle_T c_T(Te_i) c(f_p^H)\\ &+ \frac{1}{4}\langle S_T f_p^H, f_q^H\rangle_T c(f_p^H) c(f_q^H)\end{aligned} \tag{2.59}$$

is a Clifford connection associated with ∇_T^{TZ}, where $\langle\cdot,\cdot\rangle_T = g_T^{TZ}(\cdot,\cdot)$ (cf. [25, (4.3)]). Thus we get a rescaled equivariant geometric family

$$\mathcal{F}_{Z,T} := (U, \mathcal{E}_Z, T_{\pi_Z}^H U, g_T^{TZ}, h^{\mathcal{E}_Z}, \nabla_T^{\mathcal{E}_Z}) \tag{2.60}$$

over B. We write $\mathcal{F}_Z = \mathcal{F}_{Z,1}$.

Let $\mathcal{A}_M$ be a perturbation operator with respect to $D(\mathcal{F}_M)$. Then $\mathcal{A}_M$ could be extended to $1\widehat{\otimes}\mathcal{A}_M$ on $\mathscr{C}^\infty(U, \pi_Z^*\Lambda(T^*B)\widehat{\otimes}\pi_M^*\mathcal{E}_Y\widehat{\otimes}\mathcal{E}_M)$.

In [24, Lemma 2.15], we prove that for any compact submanifold K of B, there exists $T_0 > 0$ such that for $T \geq T_0$, $1\widehat{\otimes}T\mathcal{A}_M$ is a perturbation operator with respect to $D(\mathcal{F}_{Z,T})$ over K.

The following theorem is the Clifford module version of [24, Lemma 2.16], which is related to [17, Theorem 0.1], [26, Theorem 3.1], [14, Theorem 5.11] and [25, Theorem 3.4].

Theorem 2.7 *For any compact submanifold K of B, there exists $T_0 > 0$ such that for $T \geq T_0$, modulo exact forms on B, over K, we have*

$$\begin{aligned}\widetilde{\eta}_g(\mathcal{F}_{Z,T}, 1\widehat{\otimes}T\mathcal{A}_M) = &\int_{Y^g} \widehat{\mathrm{A}}_g(TY, \nabla^{TY})\,\mathrm{ch}_g(\mathcal{E}_Y/\mathcal{S}, \nabla^{\mathcal{E}_Y})\,\widetilde{\eta}_g(\mathcal{F}_M, \mathcal{A}_M)\\ &- \int_{Z^g} \left(\widetilde{\widehat{\mathrm{A}}}_g \cdot \widetilde{\mathrm{ch}}_g\right)\left(\nabla_T^{TZ}, \nabla^{TY,TM}, \nabla_T^{\mathcal{E}_Z}, {}^0\nabla^{\mathcal{E}_Z}\right).\end{aligned} \tag{2.61}$$

3 Embedding of equivariant eta forms

In this section, we state our main result and give an application in equivariant Atiyah–Hirzebruch direct image. In Sect. 3.1, we describe the geometry of the embedding of submersions. In Sect. 3.2, we explain our assumptions on the embedding of the geometric families. In Sect. 3.3, we introduce the equivariant Atiyah–Hirzebruch direct image. In Sect. 3.4, we state our main result.

3.1 Embedding of submersions

In this subsection, we introduce the embedding of submersions following [8, Section 1] and [12].

Let $i : W \to V$ be an embedding of smooth oriented manifolds. Let $\pi_V : V \to B$ be a submersion of smooth oriented manifolds with compact fibres X, whose restriction $\pi_W : W \to B$ is a smooth submersion with compact fibres Y.

Thus, we have the diagram of maps

$$\begin{array}{ccccc} Y & \longrightarrow & W & & \\ {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & \overset{\pi_W}{\searrow} & \\ X & \longrightarrow & V & \overset{\pi_V}{\longrightarrow} & B. \end{array}$$

In general, B, V, W are not connected. We simply assume that B and V are connected. For any connected component W_α of W, we assume that $\dim V - \dim W_\alpha$ is even. To simplify the notations, we usually denote the connected component by W when there is no confusion.

Let $TX = TV/B$, $TY = TW/B$ be the relative tangent bundles to the fibres X, Y. Let $T^H V$ be a smooth subbundle of TV such that

$$TV = T^H V \oplus TX. \tag{3.1}$$

Let $N_{W/V}$ be the normal bundle to W in V, let $N_{Y/X}$ be the normal bundle to Y in X. Clearly $N_{W/V} = N_{Y/X}$. Let $\widetilde{N}_{Y/X}$ be a smooth subbundle of $TX|_W$ such that

$$TX|_W = TY \oplus \widetilde{N}_{Y/X}. \tag{3.2}$$

Clearly,

$$T^H V \simeq \pi_V^* TB, \quad \widetilde{N}_{Y/X} \simeq N_{Y/X}. \tag{3.3}$$

By (3.1) and (3.2), we get

$$TV|_W = T^H V|_W \oplus TY \oplus \widetilde{N}_{Y/X}. \tag{3.4}$$

By (3.4), there is a well-defined morphism

$$\frac{TW}{TY} \to T^H V|_W \oplus \widetilde{N}_{Y/X} \tag{3.5}$$

and this morphism maps TW/TY into a subbundle of TW. Let $T^H W$ be the subbundle of TW which is the image of TW/TY by the morphism (3.5). Clearly,

$$TW = T^H W \oplus TY. \tag{3.6}$$

Note that $T^H W$ depends on the choice of $\widetilde{N}_{Y/X}$. In general, the subbundle $T^H W$ is not equal to $T^H V|_W$.

Let g^{TV} be a metric on TV. Let g^{TW} be the induced metric on TW. Let g^{TX}, g^{TY} be the induced metrics on TX, TY. Note that even if g^{TV} is of the type as in (2.12), in general, g^{TW} is not of this type.

We identity $N_{Y/X}$ with the orthogonal bundle $\widetilde{N}_{Y/X}$ to TY in $TX|_W$ with respect to $g^{TX}|_W$. Let $g^{N_{Y/X}}$ be the induced metric on $N_{Y/X}$. On W, we have

$$TX|_W = TY \oplus N_{Y/X}. \tag{3.7}$$

To the pairs $(T^H_{\pi_V}V, g^{TX})$ and $(T^H_{\pi_W}W, g^{TY})$, we can associate the objects that we construct in (2.13) and (2.15). In particular, TX, TY are now equipped with connections ∇^{TX}, ∇^{TY} which preserve the metrics g^{TX}, g^{TY} respectively.

Let P^{TY}, $P^{N_{Y/X}}$ be the orthogonal projections $TX|_W \to TY$, $TX|_W \to N_{Y/X}$. By [8, Theorem 1.9], we have

$$\nabla^{TY} = P^{TY}\nabla^{TX|_Y}. \tag{3.8}$$

Let

$$\nabla^{N_{Y/X}} = P^{N_{Y/X}}\nabla^{TX} \tag{3.9}$$

be the connection on $N_{Y/X}$. Then $\nabla^{N_{Y/X}}$ preserves the metric $g^{N_{Y/X}}$. Put

$$\nabla^{TY,N_{Y/X}} = \nabla^{TY} \oplus \nabla^{N_{Y/X}}. \tag{3.10}$$

Then $\nabla^{TY,N_{Y/X}}$ is a Euclidean connection on $TX|_W = TY \oplus N_{Y/X}$.

Let G be a compact Lie group. We assume that W, V and B are G-manifolds and the G-action commutes with the embedding and π_V. Obviously, the group action commutes with π_W. We assume that G acts trivially on B. We assume that the group action preserves the splittings (3.1) and (3.6) and all metrics and connections are G-invariant.

Let W^g, V^g be the fixed point sets of W, V for $g \in G$. Then $\pi_W|_{W^g} : W^g \to B$ and $\pi_V|_{V^g} : V^g \to B$ are submersions with compact fibres Y^g and X^g. We assume that TY^g and TX^g are all oriented as the beginning of Sect. 2.3.

Remark 3.1 (cf. [8, Section 7.5]) Given a G-equivariant pair $(T^H_{\pi_W}W, g^{TY})$, we could take metrics g^{TB} and g^{TW} on TB and TW such that $g^{TW} = \pi_W^* g^{TB} \oplus g^{TY}$. Let g^N be a G-invariant metric on $N_{Y/X}$. Let ∇^N be a G-invariant Euclidean connection on $(N_{Y/X}, g^N)$ and T^HN be the horizontal subbundle associated with the fibration $\pi_N : N_{Y/X} \to W$ and ∇^N. We take $g^{TN} = \pi_N^* g^{TW} \oplus g^N$ for $TN = T^HN \oplus N$. Since W intersects X orthogonally, we could take a horizontal subbundle $T^H_{\pi_V}V$ over V such that $T^H_{\pi_V}V|_W = T^H_{\pi_W}W$. Using the partition of unity argument, we could construct G-invariant metrics g^{TX}, g^{TV} on TX, TV such that $g^{TV} = \pi_V^* g^{TB} \oplus g^{TX}$ and W is a totally geodesic submanifold of V. In this case, for any $b \in B$, the fibre Y_b is a totally geodesic submanifold of X_b. It means that $\nabla^{TX|_W} = \nabla^{TY,N_{Y/X}}$.

By Remark 3.1, in this paper, we will always assume that the pairs $(T^H_{\pi_W}W, g^{TY})$ and $(T^H_{\pi_V}V, g^{TX})$ satisfy the conditions that

$$T^H_{\pi_V}V|_W = T^H_{\pi_W}W, \quad \nabla^{TX|_W} = \nabla^{TY,N_{Y/X}}. \tag{3.11}$$

3.2 Embedding of the geometric families

In this subsection, we state our assumptions on the embedding of the geometric families, which is the equivariant family case of the assumptions in [13, Section 1 b)].

Let $\mathcal{F}_Y := (W, \mathcal{E}_Y, T^H_{\pi_W}W, g^{TY}, h^{\mathcal{E}_Y}, \nabla^{\mathcal{E}_Y})$ and $\mathcal{F}_X := (V, \mathcal{E}_X, T^H_{\pi_V}V, g^{TX}, h^{\mathcal{E}_X}, \nabla^{\mathcal{E}_X})$ be two equivariant geometric families over B such that the pairs $(T^H_{\pi_W}W, g^{TY})$ and $(T^H_{\pi_V}V, g^{TX})$ satisfy (3.11). For simplicity, we assume that for any connected component W_α, $\tau_\alpha^{\mathcal{E}_Y/\mathcal{S}} \equiv 1$ on $\mathcal{E}_Y$.
Assume that $(N_{Y/X}, g^{N_{Y/X}})$ *has an equivariant Spin^c structure*. Then there exists an equivariant complex line bundle L_N (cf. [23, Appendix D]) such that $w_2(N_{Y/X}) = c_1(L_N) \bmod 2$,

where w_2 is the second Stiefel–Whitney class and c_1 is the first Chern class. Let $\mathcal{S}(N_{Y/X}, L_N)$ be the spinor bundle for L_N which locally may be written as

$$\mathcal{S}(N_{Y/X}, L_N) = \mathcal{S}_0(N_{Y/X}) \otimes L_N^{1/2}, \tag{3.12}$$

where $\mathcal{S}_0(N_{Y/X})$ is the spinor bundle for the (possibly non-existent) spin structure on $N_{Y/X}$ and $L_N^{1/2}$ is the (possibly non-existent) square root of L_N. Then the G-actions on $N_{Y/X}$ and L_N lift to $\mathcal{S}(N_{Y/X}, L_N)$. For simplicity, we usually simply denote the spinor bundle by $\mathcal{S}_N$.

Let h^L be a G-invariant Hermitian metric on L_N. Let ∇^L be a G-invariant Hermitian connection on (L_N, h^L). Let $h^{\mathcal{S}_N}$ be the equivariant Hermitian metric on $\mathcal{S}_N$ induced by $g^{N_{Y/X}}$ and h^L. Let $\nabla^{\mathcal{S}_N}$ be the equivariant Hermitian connection on $\mathcal{S}_N$ induced by $\nabla^{N_{Y/X}}$ and ∇^L.

From (2.19), the bundle $\mathrm{End}_{C(TX)}(\mathcal{E}_X)$ is naturally $\mathbb{Z}_2$-graded with respect to $\tau^{\mathcal{E}_X/S}$. Let $\mathcal{V}$ be a smooth self-adjoint section of $\mathrm{End}_{C(TX)}(\mathcal{E}_X)$ such that it exchanges this $\mathbb{Z}_2$-grading and commutes with the G-action. Then $\mathcal{V}$ acts on $\pi_V^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}_X$ in the same way as the perturbation operator $\mathcal{A}$ in (2.42).

We assume that on $V\backslash W$, $\mathcal{V}$ is invertible, and that on W, $\ker\mathcal{V}$ *has locally constant nonzero dimension, so that* $\ker\mathcal{V}$ *is a nonzero smooth $\mathbb{Z}_2$-graded G-equivariant vector subbundle of* $\mathcal{E}_X|_W$. Let $h^{\ker\mathcal{V}}$ be the metric on $\ker\mathcal{V}$ induced by the metric $h^{\mathcal{E}_X|_W}$. Let $P^{\ker\mathcal{V}}$ be the orthogonal projection operator from $\mathcal{E}_X|_W$ onto $\ker\mathcal{V}$.

For $y \in W$, $U \in T_yX$, let $\partial_U\mathcal{V}(y)$ be the derivative of $\mathcal{V}$ with respect to U in any given smooth trivialization of $\mathcal{E}_X$ near $y \in W$. One then verifies that $P^{\ker\mathcal{V}}\partial_U\mathcal{V}(y)P^{\ker\mathcal{V}}$ does not depend on the trivialization, and only depends on the image Z of $U \in T_yX$ in $N_{Y/X}$. From now on, we will write $\dot{\partial}_Z(\mathcal{V})(y)$ instead of $P^{\ker\mathcal{V}}\partial_U\mathcal{V}(y)P^{\ker\mathcal{V}}$. Then one verifies that $\dot{\partial}_Z(\mathcal{V})(y)$ is a self-adjoint element of $\mathrm{End}(\ker\mathcal{V})$ and exchanges the $\mathbb{Z}_2$-grading.

If $Z \in N_{Y/X}$, let $\tilde{c}(Z) \in \mathrm{End}(\mathcal{S}_N^*)$ be the transpose of $c(Z)$ acting on $\mathcal{S}_N$.

Denote by $N_{\mathbb{C}}^* = N_{Y/X}^* \otimes_{\mathbb{R}} \mathbb{C}$. Since $L_N \otimes L_N^*$ is an equivariant trivial bundle and since $\dim N_{Y/X}$ is even, we have $\Lambda(N_{\mathbb{C}}^*) \simeq \mathcal{S}_N\widehat{\otimes}\mathcal{S}_N^*$. We equip $\Lambda(N_{\mathbb{C}}^*)\widehat{\otimes}\mathcal{E}_Y$ with the induced metric $h^{\Lambda(N_{\mathbb{C}}^*)\widehat{\otimes}\mathcal{E}_Y}$. For $Z \in N_{Y/X}$, $\tilde{c}(Z)$ acts on $\mathcal{S}_N\widehat{\otimes}\mathcal{S}_N^*\widehat{\otimes}\mathcal{E}_Y$ like $1 \otimes \tilde{c}(Z) \otimes 1$.

Fundamental assumption Let $\pi_N : N_{Y/X} \to W$ be the projection. Over the total space $N_{Y/X}$, we have the equivariant identification

$$\begin{aligned}&\left(\pi_N^*\ker\mathcal{V}, \pi_N^*h^{\ker\mathcal{V}}, \dot{\partial}_Z(\mathcal{V})(y)\right)\\ &\quad\simeq \left(\pi_N^*(\Lambda(N_{\mathbb{C}}^*)\widehat{\otimes}\mathcal{E}_Y), \pi_N^*h^{\Lambda(N_{\mathbb{C}}^*)\widehat{\otimes}\mathcal{E}_Y}, \sqrt{-1}\tilde{c}(Z)\right).\end{aligned} \tag{3.13}$$

Let $\nabla^{\ker\mathcal{V}}$ be the equivariant Hermitian connection on $\ker\mathcal{V}$,

$$\nabla^{\ker\mathcal{V}} = P^{\ker\mathcal{V}}\nabla^{\mathcal{E}_X|_W}P^{\ker\mathcal{V}}. \tag{3.14}$$

We make the assumption that under the identification (3.13),

$$\nabla^{\ker\mathcal{V}} = \nabla^{\Lambda(N_{\mathbb{C}}^*)\widehat{\otimes}\mathcal{E}_Y}. \tag{3.15}$$

3.3 Atiyah–Hirzebruch direct image

In this subsection, we introduce an important example of the embedding of equivariant geometric families satisfying the fundamental assumption: the equivariant version of the Atiyah–Hirzebruch direct image [1,21]. We assume that the base space B is compact and adopt the notations and the assumptions in Sect. 3.1 in this subsection.

We further assume that TY and TX have equivariant Spin^c structures. Then there exist equivariant complex line bundles L_Y and L_X over W and V such that $w_2(TY) = c_1(L_Y) \mod 2$ and $w_2(TX) = c_1(L_X) \mod 2$. Then from the splitting (3.7), the equivariant vector bundle $N_{Y/X}$ over W has an equivariant Spin^c structure with associated equivariant line bundle $L_N := L_X \otimes L_Y^{-1}$. Let h^{L_Y}, h^{L_X} be G-invariant Hermitian metrics on L_Y, L_X and ∇^{L_Y}, ∇^{L_X} be G-invariant Hermitian connections on (L_Y, h^{L_Y}), (L_X, h^{L_X}). Let h^{L_N} and ∇^{L_N} be metric and connection on L_N induced by h^{L_Y}, h^{L_X} and ∇^{L_Y}, ∇^{L_X}. Let $\mathcal{S}(TY, L_Y)$, $\mathcal{S}(TX, L_X)$ and $\mathcal{S}(N_{Y/X}, L_N)$ be the spinor bundles for (TY, L_Y), (TX, L_X) and $(N_{Y/X}, L_N)$, which we will simply denote by $\mathcal{S}_Y$, $\mathcal{S}_X$ and $\mathcal{S}_N$. Then these spinors are G-equivariant vector bundles. Furthermore, $\mathcal{S}_X|_W = \mathcal{S}_Y \widehat{\otimes} \mathcal{S}_N$. Since $\dim N_{Y/X} = \dim V - \dim W$ is even, the spinor $\mathcal{S}_N$ is $\mathbb{Z}_2$-graded.

Recall that $\{W_\alpha\}_{\alpha=1,\ldots,k}$ are the connected components of W. Let (μ, h^μ) be a G-equivariant Hermitian vector bundle over W with a G-invariant Hermitian connection ∇^μ. In the followings, we will describe a geometric realization of the Atiyah–Hirzebruch direct image $i![\mu] \in \widetilde{K}_G^0(V)$ as in [1,21]. We denote by μ_α the restriction of μ on W_α.

For any $r > 0$, set $N_{\alpha,r} := \{Z \in N_{Y_\alpha/X} : |Z| < r\}$. Then there is $\varepsilon_0 > 0$ such that the map $(y, Z) \in N_{Y_\alpha/X} \to \exp_y^V(Z)$ defines a diffeomorphism of $N_{\alpha,2\varepsilon_0}$ on an open G-equivariant tubular neighbourhood of W_α in V for any α. Without confusion we will also regard $N_{\alpha,2\varepsilon_0}$ as the open G-equivariant tubular neighbourhood of W in V. We choose $\varepsilon_0 > 0$ small enough such that for any $1 \le \alpha \neq \beta \le k$, $N_{\alpha,2\varepsilon_0} \cap N_{\beta,2\varepsilon_0} = \emptyset$.

Let $\pi_\alpha : N_{Y_\alpha/X} \to W_\alpha$ denote the projection of the normal bundle $N_{Y_\alpha/X}$ on W_α. For $Z \in N_{Y_\alpha/X}$, let $\tilde{c}(Z) \in \mathrm{End}(\mathcal{S}_{N_\alpha}^*)$ be the transpose of $c(Z)$ acting on $\mathcal{S}_{N_\alpha}$. Let $\pi_\alpha^*(\mathcal{S}_{N_\alpha}^*)$ be the pull back bundle of $\mathcal{S}_{N_\alpha}^*$ over $N_{Y_\alpha/X}$. For any $Z \in N_{Y_\alpha/X}$ with $Z \neq 0$, $\tilde{c}(Z) : \pi_\alpha^*(\mathcal{S}_{N_\alpha,\pm}^*)|_Z \to \pi_\alpha^*(\mathcal{S}_{N_\alpha,\mp}^*)|_Z$ is an equivariant isomorphism at Z.

From the equivariant Serre–Swan theorem [29, Proposition 2.4], there exists a G-equivariant Hermitian vector bundle (E_α, h^{E_α}) such that $\mathcal{S}_{N_\alpha,-}^* \otimes \mu_\alpha \oplus E_\alpha$ is a G-equivariant trivial complex vector bundle over W_α. Then

$$\tilde{c}(Z) \oplus \pi_\alpha^* \mathrm{Id}_{E_\alpha} : \pi_\alpha^*(\mathcal{S}_{N_\alpha,+}^* \otimes \mu_\alpha \oplus E_\alpha) \to \pi_\alpha^*(\mathcal{S}_{N_\alpha,-}^* \otimes \mu_\alpha \oplus E_\alpha) \tag{3.16}$$

induces a G-equivariant isomorphism between two equivariant trivial vector bundles over $N_{\alpha,2\varepsilon_0} \setminus W_\alpha$.

By adding the equivariant trivial bundles, we could assume that for any $1 \le \alpha \neq \beta \le k$, $\dim(\mathcal{S}_{N_\alpha,\pm}^* \otimes \mu_\alpha \oplus E_\alpha) = \dim(\mathcal{S}_{N_\beta,\pm}^* \otimes \mu_\beta \oplus E_\beta)$. Clearly, $\{\pi_\alpha^*(\mathcal{S}_{N_\alpha,\pm}^* \otimes \mu_\alpha \oplus E_\alpha)|_{\partial N_{\alpha,2\varepsilon_0}}\}_{\alpha=1,\ldots,k}$ extend smoothly to two equivariant trivial complex vector bundles over $V \setminus \cup_{1 \le \alpha \le k} N_{\alpha,2\varepsilon_0}$.

In summary, what we get is a $\mathbb{Z}_2$-graded Hermitian vector bundle (ξ, h^ξ) such that

$$\begin{aligned} \xi_\pm|_{N_{\alpha,\varepsilon_0}} &= \pi_\alpha^*(\mathcal{S}_{N_\alpha,\pm}^* \otimes \mu_\alpha \oplus E_\alpha)|_{N_{\alpha,\varepsilon_0}}, \\ h^{\xi_\pm}|_{N_{\alpha,\varepsilon_0}} &= \pi_\alpha^* \left(h^{\mathcal{S}_{N_\alpha,\pm}^* \otimes \mu_\alpha} \oplus h^{E_\alpha} \right)\Big|_{N_{\alpha,\varepsilon_0}}, \end{aligned} \tag{3.17}$$

where $h^{\mathcal{S}_{N_\alpha,\pm}^* \otimes \mu_\alpha}$ is the equivariant Hermitian metric on $\mathcal{S}_{N_\alpha,\pm}^* \otimes \mu_\alpha$ induced by g^{N_α}, $h^{L_{N_\alpha}}$ and h^{μ_α}. Let ∇^{E_α} be a G-invariant Hermitian connection on (E_α, h^{E_α}). We can also get a G-invariant $\mathbb{Z}_2$-graded Hermitian connection $\nabla^\xi = \nabla^{\xi_+} \oplus \nabla^{\xi_-}$ on $\xi = \xi_+ \oplus \xi_-$ over V such that

$$\nabla^{\xi_\pm}|_{N_{\alpha,\varepsilon_0}} = \pi_\alpha^* \left(\nabla^{\mathcal{S}_{N_\alpha,\pm}^* \otimes \mu_\alpha} \oplus \nabla^{E_\alpha} \right), \tag{3.18}$$

where $\nabla^{\mathcal{S}^*_{N_\alpha,\pm}\otimes\mu_\alpha}$ is the equivariant Hermitian connection on $\mathcal{S}^*_{N_\alpha,\pm}\otimes\mu_\alpha$ induced by ∇^{N_α}, $\nabla^{L_{N_\alpha}}$ and ∇^{μ_α}.

It is easy to see that there exists an equivariant self-adjoint automorphism $\mathcal{V}$ of $\mathcal{S}_X\widehat{\otimes}\xi$, which exchanges the $\mathbb{Z}_2$-grading of ξ, such that

$$\mathcal{V}|_{N_{\alpha,\varepsilon_0}} = \mathrm{Id}_{\mathcal{S}_X}\,\widehat{\otimes}\left(\sqrt{-1}\,\tilde{c}(Z)\oplus\pi^*\mathrm{Id}_{E_\alpha}\right). \tag{3.19}$$

From the construction above, we could see that $\mathcal{V}$ is invertible on $V\backslash W$ and

$$(\ker\mathcal{V})|_W = \mathcal{S}_X|_W\widehat{\otimes}\mathcal{S}^*_N\otimes\mu = \mathcal{S}_Y\widehat{\otimes}\mathcal{S}_N\widehat{\otimes}\mathcal{S}^*_N\otimes\mu = \mathcal{S}_Y\widehat{\otimes}\Lambda(N^*_{\mathbb{C}})\otimes\mu \tag{3.20}$$

is an equivariant vector bundle over W. Let $P^{\ker\mathcal{V}}$ be the orthogonal projection from $\mathcal{S}_X\widehat{\otimes}\xi|_W$ onto $\ker\mathcal{V}$ and $\nabla^{\ker\mathcal{V}} = P^{\ker\mathcal{V}}\nabla^{\mathcal{S}_X\widehat{\otimes}\xi|_W}P^{\ker\mathcal{V}}$. From (3.11), we have

$$\nabla^{\ker\mathcal{V}} = \nabla^{\mathcal{S}_Y\widehat{\otimes}\Lambda(N^*_{\mathbb{C}})\otimes\mu}. \tag{3.21}$$

Here $[\xi_+]-[\xi_-]\in\widetilde{K}^0_G(V)$ is an equivariant version of the Atiyah–Hirzebruch direct image $i![\mu]$ in [1]. In this construction, let $\mathcal{E}_Y=\mathcal{S}_Y\otimes\mu$ and $\mathcal{E}_{X,\pm}=\mathcal{S}_X\widehat{\otimes}\xi_\pm$. Then it satisfies all assumptions in Sect. 3.2.

3.4 Main result

In this subsection, we state our main result.

Let $\mathcal{F}_Y$ and $\mathcal{F}_X$ be the equivariant geometric families satisfying the assumptions in Sect. 3.2.

For $T\geq 0$, let $\nabla^{\mathcal{E}_X,T}$ be the superconnection on $\mathcal{E}_X$ given by

$$\nabla^{\mathcal{E}_X,T} = \nabla^{\mathcal{E}_X}+\sqrt{T}\mathcal{V}. \tag{3.22}$$

Let $R_T^{\mathcal{E}_X/\mathcal{S}}$ be the twisting curvature of $\nabla^{\mathcal{E}_X,T}$ as in (2.32). Let $\dim(N_{X^g/X})=\ell_2$. For $T>0$, by [28] and (2.31), we have the equivariant version of [13, (1.17)]:

$$\begin{aligned}&\frac{\partial}{\partial T}\mathrm{Tr}^{\mathcal{E}_X/\mathcal{S}}\left[\sigma_{\ell_2}(g^{\mathcal{E}_X})\exp\left(-R_T^{\mathcal{E}_X/\mathcal{S}}|_{V^g}\right)\right]\\ &\quad = -d\,\mathrm{Tr}^{\mathcal{E}_X/\mathcal{S}}\left[\sigma_{\ell_2}(g^{\mathcal{E}_X})\frac{\mathcal{V}|_{V^g}}{2\sqrt{T}}\exp\left(-R_T^{\mathcal{E}_X/\mathcal{S}}|_{V^g}\right)\right].\end{aligned} \tag{3.23}$$

Recall that ψ is defined in (2.37). The proof of the following theorem is the same as those of [7, Theorem 6.3] and [13, Theorem 1.2].

Theorem 3.2 *For any compact set $K\subset V^g$, there exists $C>0$, such that if $\omega\in\Omega^*(V^g)$ has support in K,*

$$\begin{aligned}&\left|\int_{X^g}\omega\cdot\frac{2^{\ell_2/2}}{\det^{1/2}(1-g|_{N_{X^g/X}})}\psi_{V^g}\,\mathrm{Tr}^{\mathcal{E}_X/\mathcal{S}}\left[\sigma_{\ell_2}(g^{\mathcal{E}_X})\exp\left(-R_T^{\mathcal{E}_X/\mathcal{S}}|_{V^g}\right)\right]\right.\\ &\left.\quad-\int_{Y^g}\omega\cdot\widehat{\mathrm{A}}_g^{-1}(N_{Y/X},\nabla^{N_{Y/X}})\,\mathrm{ch}_g(\mathcal{E}_Y/\mathcal{S},\nabla^{\mathcal{E}_Y})\right|\leq\frac{C}{\sqrt{T}}\|\omega\|_{\mathscr{C}^1(K)},\end{aligned} \tag{3.24}$$

and

$$\left|\int_{X^g}\omega\cdot\psi_{V^g}\,\mathrm{Tr}^{\mathcal{E}_X/\mathcal{S}}\left[\sigma_{\ell_2}(g^{\mathcal{E}_X})\frac{\mathcal{V}|_{V^g}}{2\sqrt{T}}\exp\left(-R_T^{\mathcal{E}_X/\mathcal{S}}|_{V^g}\right)\right]\right|\leq\frac{C}{T^{3/2}}\|\omega\|_{\mathscr{C}^1(K)}. \tag{3.25}$$

Now we could extend the Bismut–Zhang current in [13, Definition 1.3] to the equivariant case.

Definition 3.3 The equivariant Bismut–Zhang current $\gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X)$ over V^g is defined by

$$\gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X) = \frac{1}{2\sqrt{\pi}\sqrt{-1}} \cdot \frac{2^{\ell_2/2}}{\det^{1/2}(1 - g|_{N_{X^g/X}})} \cdot \int_0^\infty \psi_{V^g} \operatorname{Tr}^{\mathcal{E}_X/\mathcal{S}} \left[\sigma_{\ell_2}(g^{\mathcal{E}_X})\mathcal{V}|_{V^g} \exp\left(-R_T^{\mathcal{E}_X/\mathcal{S}}|_{V^g}\right)\right] \frac{dT}{2\sqrt{T}}. \tag{3.26}$$

By Theorem 3.2, the current $\gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X)$ is well-defined.

Let δ_{W^g} be the current of integration over the submanifold W^g in V^g. By integrating (3.23) and using Theorem 3.2, we have the following equivariant extension of [13, Theorem 1.4].

Theorem 3.4 *The following equation of currents on V^g holds*

$$d\gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X) = \operatorname{ch}_g(\mathcal{E}_X/\mathcal{S}, \nabla^{\mathcal{E}_X}) - \widehat{\mathrm{A}}_g^{-1}(N_{Y/X}, \nabla^{N_{Y/X}}) \operatorname{ch}_g(\mathcal{E}_Y/\mathcal{S}, \nabla^{\mathcal{E}_Y})\delta_{W^g}. \tag{3.27}$$

Remark 3.5 Similarly as in [13], the wave front set $\mathrm{WF}(\gamma_g^X)$ of the current γ_g^X is included in $N^*_{W^g/V^g}$ and $\gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X)$ is a locally integrable current.

Proposition 3.6 *Let $\mathcal{A}_Y$ be a perturbation operator with respect to $D(\mathcal{F}_Y)$. Then we could construct a family of bounded pseudodifferential operator $\mathcal{A}_{T,Y}$ on $\mathcal{F}_X$, depending continuously on $T \geq 1$, such that the norm of $\mathcal{A}_{T,Y}$ is the same as that of $\mathcal{A}_Y$ for any $T \geq 1$ and for any compact submanifold K of B, there exists $T_0 \geq 1$ depending on K such that $T\mathcal{V} + \mathcal{A}_{T,Y}$ is the perturbation operator with respect to $D(\mathcal{F}_X)$ over K for $T \geq T_0$.*

Proof Following the arguments in [11, Section 8, 9] and [13, Section 4b)] word by word, we could construct a smooth family of equivariant linear isometric embeddings

$$J_{T,b}\colon L^2(Y_b, \mathcal{E}_Y|_{Y_b}) \to L^2(X_b, \mathcal{E}_X|_{X_b}) \tag{3.28}$$

for $b \in B$, as in [8, Definition 9.12].

Let $\mathbb{E}_{T,b}$ be the image of $L^2(Y_b, \mathcal{E}_Y|_{Y_b})$ in $L^2(X_b, \mathcal{E}_X|_{X_b})$ by $J_{T,b}$. Let $\mathbb{E}_{T,b}^\perp$ be the orthogonal space to $\mathbb{E}_{T,b}$ in $L^2(X_b, \mathcal{E}_X|_{X_b})$. Since $J_{T,b}$ is an isometric embedding, $J_{T,b}\colon L^2(Y_b, \mathcal{E}_Y|_{Y_b}) \to \mathbb{E}_T$ is invertible. We extend the domain of $J_{T,b}^{-1}$ to $L^2(X_b, \mathcal{E}_X|_{X_b})$ linearly such that it vanishes on $\mathbb{E}_{T,b}^\perp$.

Let $\mathcal{A}_{T,Y} = \{\mathcal{A}_{T,Y,b}\}_{b\in B}$ be the family of bounded pseudodifferential operators

$$\mathcal{A}_{T,Y,b} := J_{T,b}\mathcal{A}_{Y,b}J_{T,b}^{-1} : L^2(X_b, \mathcal{E}_X|_{X_b}) \to L^2(X_b, \mathcal{E}_X|_{X_b}). \tag{3.29}$$

Then $\mathcal{A}_{T,Y}$ is a smooth family of equivariant self-adjoint operators. From the definition of the perturbation operator $\mathcal{A}_Y$, we see that $\mathcal{A}_{T,Y}$ commutes (resp. anti-commutes) with the $\mathbb{Z}_2$-grading $\tau^{\mathcal{E}}$ of $\mathcal{E}_X$ when the fibres are odd (resp. even) dimensional. Since J_T is isometric, the L^2-norm of $\mathcal{A}_{T,Y}$ is the same as that of $\mathcal{A}_Y$.

Since J_T is continuous with respect to T, so is the operator $\mathcal{A}_{T,Y}$. We only need to prove that $D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y}$ over K is invertible for T large enough.

Over a compact submanifold K of B, the same estimates of $D(\mathcal{F}_X) + T\mathcal{V}$ as [11, Theorem 9.8, 9.10, 9.11] hold. Since $D(\mathcal{F}_Y)+\mathcal{A}_Y$ is invertible, the arguments in [11, Section 9], in which we replace $D(\mathcal{F}_Y)$ and $D(\mathcal{F}_X)+T\mathcal{V}$ by $D(\mathcal{F}_Y)+\mathcal{A}_Y$ and $D(\mathcal{F}_X)+T\mathcal{V}+\mathcal{A}_{T,Y}$, imply that there exists $T_0 \geq 1$, depending on K, such that for any $T \geq T_0$, $D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y}$ is invertible. Moreover, the absolutely value of the spectrum of $D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y}$ has a uniformly positive lower bound for $T \geq T_0$.

The proof of our proposition is completed. □

Now we state our main result of this paper.

Theorem 3.7 *Let $\mathcal{A}_Y$ and $\mathcal{A}_X$ be the perturbation operators with respect to $D(\mathcal{F}_Y)$ and $D(\mathcal{F}_X)$ respectively. Let $\mathcal{A}_{T,Y}$ be the operator constructed in Proposition 3.6. Then for any compact submanifold K of B, there exists $T_0 > 2$ depending on K such that for any $T \geq T_0$, modulo exact forms on B, over K, we have*

$$\tilde{\eta}_g(\mathcal{F}_X, \mathcal{A}_X) = \tilde{\eta}_g(\mathcal{F}_Y, \mathcal{A}_Y) + \int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX})\, \gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X) + \mathrm{ch}_g(\mathrm{sf}_G\{D(\mathcal{F}_X) + \mathcal{A}_X, D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y}\}). \tag{3.30}$$

Observe that since we only need to prove (3.30) over a compact submanifold, in the proof of Theorem 3.7, we may assume that B is compact.

If the base space is a point, and if Y and X are odd dimensional spin manifolds, then there exist equivariant complex vector bundles μ and $\xi_\pm$ such that $\mathcal{E}_Y = \mathcal{S}_Y \otimes \mu$ and $\mathcal{E}_{X,\pm} = \mathcal{S}_X \widehat{\otimes} \xi_\pm$. The following corollary is a direct consequence of Theorem 3.7.

Corollary 3.8 *There exists $x \in R(G)$, the representation ring of G, such that*

$$\overline{\eta}_g(X, \xi_+) - \overline{\eta}_g(X, \xi_-) = \overline{\eta}_g(Y, \mu) + \int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX})\, \gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X) + \chi_g(x). \tag{3.31}$$

Here x could be written as an equivariant spectral flow, $\chi_g(x)$ is the character of g on x and $\overline{\eta}_g$ is the equivariant reduced APS eta invariant.

When $g = 1$, Corollary 3.8 is the modification of the Bismut–Zhang embedding formula by expressing the mod $\mathbb{Z}$ term as a spectral flow. Note that in [19, Theorem 4.1], the authors give an index interpretation of the mod $\mathbb{Z}$ term of the embedding formula when the manifolds are the boundaries. It is also interesting to look for the equivariant family extension of that formula.

Corollary 3.9 *Let X be an odd-dimensional compact G-equivariant Spin^c manifold. For $g \in G$, let (μ, h^μ) be an equivariant Hermitian vector bundle over X^g with a G-invariant Hermitian connection ∇^μ. Then there exist a $\mathbb{Z}_2$-graded equivariant Hermitian vector bundle (ξ, h^ξ) over X with a G-invariant Hermitian connection ∇^ξ and $x \in R(G)$, such that*

$$\overline{\eta}_g(X, \xi_+) - \overline{\eta}_g(X, \xi_-) = \overline{\eta}_g(X^g, \mu) + \chi_g(x). \tag{3.32}$$

Proof Note that X^g is naturally totally geodesic in X. Take (ξ, h^ξ, ∇^ξ) as the equivariant Atiyah–Hirzebruch direct image of (μ, h^μ, ∇^μ) as in Sect. 3.3. We only need to notice that $\mathcal{V}|_{X^g} = 0$ in this case. It implies that $\gamma_g^X(\mathcal{F}_{X^g}, \mathcal{F}_X) = 0$. □

Remark 3.10 Note that in [22], the authors establish an index theorem for differential K-theory. The key analytical tool is the Bismut–Zhang embedding formula of the reduced eta invariants in [13]. Using Corollary 3.8, the index theorem there could be extended to the equivariant case whenever the equivariant differential K-theory is well-defined. Using Theorem 3.7, we can also get the compatibility of the push-forward map in equivariant differential K-theory along the proper submersion and the embedding under the model of Bunck–Schick [15,16,24]. We will study these in the subsequent paper.

4 Proof of main result

In this section, we prove our main result Theorem 3.7. In Sect. 4.1, we prove Theorem 3.7 when the base space is a point using some intermediary results along the lines of [13], the proofs of which rely on almost identical arguments of [7,13]. In Sect. 4.2, we explain how to use the functoriality to reduce the proof of Theorem 3.7 to the case considered in Sect. 4.1.

4.1 Embedding of equivariant eta invariants

In this subsection, we will prove our main result when B is a point and $\dim X$ is odd. Recall that in (3.11), we have already assumed that Y is totally geodesic in X.

Theorem 4.1 *Assume that B is a point and* $\dim X$ *is odd. Then there exists $T_0 > 2$ such that for any $T \geq T_0$, we have*

$$\tilde{\eta}_g(\mathcal{F}_X, \mathcal{A}_X) = \tilde{\eta}_g(\mathcal{F}_Y, \mathcal{A}_Y) + \int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX})\, \gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X) \\ + \mathrm{ch}_g(\mathrm{sf}_G\{D(\mathcal{F}_X) + \mathcal{A}_X, D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y}\}). \tag{4.1}$$

Set

$$D_{u,T} = \sqrt{u}(D(\mathcal{F}_X) + T\mathcal{V} + \chi(\sqrt{u})((1-\chi(T))\mathcal{A}_X + \chi(T)\mathcal{A}_{T,Y})), \tag{4.2}$$

where χ is the cut-off function defined in (2.41). Let

$$\mathcal{B}_{u^2,T} = D_{u^2,T} + dT \wedge \frac{\partial}{\partial T} + du \wedge \frac{\partial}{\partial u}. \tag{4.3}$$

Definition 4.2 We define $\beta_g = du \wedge \beta_g^u + dT \wedge \beta_g^T$ to be the part of $\pi^{-1/2}\,\mathrm{Tr}_s[g \exp(-\mathcal{B}_{u^2,T}^2)]$ of degree one with respect to the coordinates (T, u), with functions $\beta_g^u, \beta_g^T : \mathbb{R}_{+,T} \times \mathbb{R}_{+,u} \to \mathbb{R}$.

From (4.3), we have

$$\begin{aligned} \beta_g^u(T,u) &= -\frac{1}{\sqrt{\pi}}\,\mathrm{Tr}_s\left[g\frac{\partial D_{u^2,T}}{\partial u}\exp(-D_{u^2,T}^2)\right], \\ \beta_g^T(T,u) &= -\frac{1}{\sqrt{\pi}}\,\mathrm{Tr}_s\left[g\frac{\partial D_{u^2,T}}{\partial T}\exp(-D_{u^2,T}^2)\right]. \end{aligned} \tag{4.4}$$

When $0 < u < 1$, $\chi(u) = 0$. In this case,

$$\begin{aligned} \beta_g^u(T,u) &= -\frac{1}{\sqrt{\pi}}\,\mathrm{Tr}_s\left[g(D(\mathcal{F}_X) + T\mathcal{V})\exp(-u^2(D(\mathcal{F}_X) + T\mathcal{V})^2)\right], \\ \beta_g^T(T,u) &= -\frac{u}{\sqrt{\pi}}\,\mathrm{Tr}_s\left[g\mathcal{V}\exp(-u^2(D(\mathcal{F}_X) + T\mathcal{V})^2)\right]. \end{aligned} \tag{4.5}$$

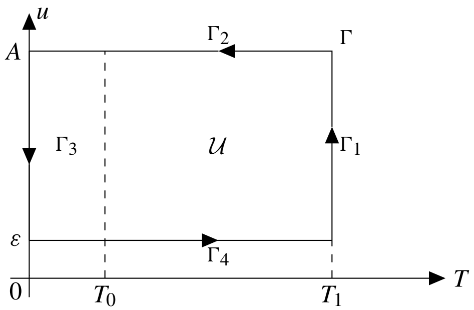
From (2.47),

$$\tilde{\eta}_g(\mathcal{F}_X, \mathcal{A}_X) = -\int_0^{+\infty} \beta_g^u(0,u)\,du. \tag{4.6}$$

As in [13, Theorem 3.4] (see also [25, Proposition 4.2]), we have

$$\left(du \wedge \frac{\partial}{\partial u} + dT \wedge \frac{\partial}{\partial T}\right)\beta_g = 0. \tag{4.7}$$

Let T_0 be the constant in Proposition 3.6. Take ε, A, T_1, $0 < \varepsilon < 1 \leq A < \infty$, $T_0 \leq T_1 < \infty$. Let $\Gamma = \Gamma_{\varepsilon,A,T_1}$ be the oriented contour in $\mathbb{R}_{+,T} \times \mathbb{R}_{+,u}$.



The contour Γ is made of four oriented pieces $\Gamma_1, \ldots, \Gamma_4$ indicated in the above picture. For $1 \leq k \leq 4$, set $I_k^0 = \int_{\Gamma_k} \beta_g$. Then by Stocks' formula and (4.7),

$$\sum_{k=1}^{4} I_k^0 = \int_{\partial\mathcal{U}} \beta_g = \int_{\mathcal{U}} \left(du \wedge \frac{\partial}{\partial u} + dT \wedge \frac{\partial}{\partial T} \right) \beta_g = 0. \tag{4.8}$$

For any $g \in G$, set

$$\beta_g^Y(u) = \frac{1}{\sqrt{\pi}} \operatorname{Tr}\left[g \exp\left(-\left(u(D(\mathcal{F}_Y) + \chi(u)\mathcal{A}_Y) + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right]^{du}. \tag{4.9}$$

Then by Definition 2.4,

$$\tilde{\eta}_g(\mathcal{F}_Y, \mathcal{A}_Y) = -\int_0^{+\infty} \beta_g^Y(u) du. \tag{4.10}$$

We now establish some estimates of β_g.

Theorem 4.3 (i) *For any $u > 0$, we have*

$$\lim_{T \to \infty} \beta_g^u(T, u) = \beta_g^Y(u). \tag{4.11}$$

(ii) *For $0 < u_1 < u_2$ fixed, there exists $C > 0$ such that, for $u \in [u_1, u_2]$, $T \geq 2$, we have*

$$|\beta_g^u(T, u)| \leq C. \tag{4.12}$$

(iii) *We have the following identity*:

$$\lim_{T \to +\infty} \int_2^{\infty} \beta_g^u(T, u) du = \int_2^{\infty} \beta_g^Y(u) du. \tag{4.13}$$

Proof If P is an operator, let $\operatorname{Spec}(P)$ be the spectrum of P. From the proof of Proposition 3.6, there exist $T_0 \geq 1$, $c > 0$, such that for $T \geq T_0$,

$$\operatorname{Spec}(D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y}) \cap [-c, c] = \emptyset. \tag{4.14}$$

Recall that $\mathbb{E}_T^0$ is the image of J_T defined in (3.28). For $\delta \in [0, 1]$, we write $D(\mathcal{F}_X) + T\mathcal{V} + \delta\mathcal{A}_{T,Y}$ in matrix form with respect to the splitting by $\mathbb{E}_T^0 \oplus \mathbb{E}_T^{0,\perp}$,

$$D(\mathcal{F}_X) + T\mathcal{V} + \delta\mathcal{A}_{T,Y} = \begin{pmatrix} A_{T,1} + \delta\mathcal{A}_{T,Y} & A_{T,2} \\ A_{T,3} & A_{T,4} \end{pmatrix}. \tag{4.15}$$

By [11, Theorem 9.8] and (3.29), as $T \to +\infty$, we have

$$J_T^{-1}(A_{T,1} + \delta \mathcal{A}_{T,Y})J_T = D(\mathcal{F}_Y) + \delta \mathcal{A}_Y + O\left(\frac{1}{\sqrt{T}}\right). \tag{4.16}$$

Set

$$\mathcal{T} := \{\delta \in [0, 1] : D(\mathcal{F}_Y) + \delta \mathcal{A}_Y \text{ is not invertible}\}. \tag{4.17}$$

Then $\mathcal{T}$ is a closed subset of $[0, 1]$.

We firstly assume that $\mathcal{T}$ is not empty. Fix $\delta_0 \in \mathcal{T}$. There exists $C(\delta_0) > 0$ such that

$$\mathrm{Spec}(D(\mathcal{F}_Y) + \delta_0 \mathcal{A}_Y) \cap [-2C(\delta_0), 2C(\delta_0)] = \{0\}. \tag{4.18}$$

Since the eigenvalues are continuous with respect to δ, there exists $\varepsilon > 0$ small enough, such that when $\delta \in (\delta_0 - \varepsilon, \delta_0 + \varepsilon)$,

$$\mathrm{Spec}(D(\mathcal{F}_Y) + \delta \mathcal{A}_Y) \cap [-C(\delta_0), C(\delta_0)] \subset (-C(\delta_0)/4, C(\delta_0)/4) \tag{4.19}$$

and

$$\begin{aligned} &\mathrm{Spec}(D(\mathcal{F}_Y) + \delta \mathcal{A}_Y) \cap (-\infty, -C(\delta_0)] \cup [C(\delta_0), +\infty) \\ &\quad \subset (-\infty, -7C(\delta_0)/4) \cup (7C(\delta_0)/4, +\infty). \end{aligned} \tag{4.20}$$

Then following the same process in [11, Section 9] and [13, Section 4 b)] by replacing $D(\mathcal{F}_Y)$ and $D(\mathcal{F}_X) + T\mathcal{V}$ by $D(\mathcal{F}_Y) + \delta \mathcal{A}_Y$ and $D(\mathcal{F}_X) + T\mathcal{V} + \delta \mathcal{A}_{T,Y}$, for $\alpha > 0$ fixed, when T is large enough, there exists $C > 0$, such that for any $\delta \in (\delta_0 - \varepsilon, \delta_0 + \varepsilon)$,

$$\begin{aligned} &\Big|\mathrm{Tr}_s\left[g(D(\mathcal{F}_X) + T\mathcal{V} + \delta \mathcal{A}_{T,Y}) \exp\left(-\alpha(D(\mathcal{F}_X) + T\mathcal{V} + \delta \mathcal{A}_{T,Y})^2\right)\right] \\ &\quad - \mathrm{Tr}\left[g(D(\mathcal{F}_Y) + \delta \mathcal{A}_Y) \exp\left(-\alpha(D(\mathcal{F}_Y) + \delta \mathcal{A}_Y)^2\right)\right]\Big| \le \frac{C}{\sqrt{T}}, \\ &\Big|\mathrm{Tr}_s\left[g \mathcal{A}_{T,Y} \exp\left(-\alpha(D(\mathcal{F}_X) + T\mathcal{V} + \delta \mathcal{A}_{T,Y})^2\right)\right] \\ &\quad - \mathrm{Tr}\left[g \mathcal{A}_Y \exp\left(-\alpha(D(\mathcal{F}_Y) + \delta \mathcal{A}_Y)^2\right)\right]\Big| \le \frac{C}{\sqrt{T}}. \end{aligned} \tag{4.21}$$

Since $\mathcal{T}$ is compact, there exists an open neighborhood $\mathcal{U}$ of $\mathcal{T}$ in $[0, 1]$ such that (4.21) hold uniformly for $\delta \in \mathcal{U}$. For $\delta \in [0, 1] \backslash \mathcal{U}$, there is a uniformly lower positive bound of the absolute value of the spectrum of $D(\mathcal{F}_Y) + \delta \mathcal{A}_Y$. So the process of [11, Section 9] also works. It means that (4.21) hold uniformly for $\delta \in [0, 1]$.

If $\mathcal{T} = \emptyset$, it means that there is a uniformly lower positive bound of the absolute value of the spectrum of $D(\mathcal{F}_Y) + \delta \mathcal{A}_Y$ for $\delta \in [0, 1]$. Thus (4.21) holds uniformly.

In summary, for $\alpha > 0$ fixed, when T is large enough, there exists $C > 0$, such that for any $\delta \in [0, 1]$, (4.21) holds.

Therefore, from Definition 4.2, (4.2), (4.3) and (4.9), we get Theorem 4.3(i) and (ii).

For $u \ge 2$ and $T \ge T_0$, from Definition 4.2, (4.2) and (4.3), we have

$$\begin{aligned} \beta_g^u(T, u) = -\frac{1}{\sqrt{\pi}} &\mathrm{Tr}_s\left[g(D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y})\right. \\ &\left.\times \exp(-u^2(D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y})^2)\right]. \end{aligned} \tag{4.22}$$

From [5, Proposition 2.37], (4.14) and (4.22), there exists $C_T > 0$, depending on $T \ge T_0$, such that for u large enough,

$$|\beta_g^u(T, u)| \le C_T \exp(-cu^2). \tag{4.23}$$

From the first inequality of (4.21) for $\delta = 1$, we see that C_T in (4.23) is uniformly bounded for $T \geq T_0$. Thus (iii) follows from (i) and the dominated convergence theorem.

The proof of our theorem is completed. □

Theorem 4.4 *Let T_0 be the constant in Proposition 3.6. When $u \to +\infty$, we have*

$$\lim_{u\to+\infty}\int_0^{T_0} \beta_g^T(T,u)dT = \mathrm{ch}_g(\mathrm{sf}_G\{D(\mathcal{F}_X)+\mathcal{A}_X, D(\mathcal{F}_X)+T_0\mathcal{V}+\mathcal{A}_{T_0,Y}\}), \tag{4.24}$$

and

$$\lim_{u\to+\infty}\int_{T_0}^{\infty} \beta_g^T(T,u)dT = 0. \tag{4.25}$$

Proof Set

$$D'_{u,T} = \sqrt{u}(D(\mathcal{F}_X)+\chi(\sqrt{u})(T\mathcal{V}+((1-\chi(T))\mathcal{A}_X+\chi(T)\mathcal{A}_{T,Y}))) \tag{4.26}$$

and

$$\beta_g^T(T,u)' = -\frac{1}{\sqrt{\pi}}\mathrm{Tr}_s\left[g\frac{\partial D'_{u^2,T}}{\partial T}\exp(-(D'_{u^2,T})^2)\right]. \tag{4.27}$$

Note that when $u > 2$,

$$\beta_g^T(T,u)' = \beta_g^T(T,u). \tag{4.28}$$

The proof of the anomaly formula Theorem 2.6 (cf. [24, Theorem 2.17]) show that

$$\begin{aligned}\lim_{u\to+\infty}\int_0^{T_0} \beta_g^T(T,u)dT &= \lim_{u\to+\infty}\int_0^{T_0} \beta_g^T(T,u)'dT\\ &= \tilde{\eta}_g(\mathcal{F}_X,\mathcal{A}_X)-\tilde{\eta}_g(\mathcal{F}_X,T_0\mathcal{V}+\mathcal{A}_{T_0,Y})\\ &= \mathrm{ch}_g(\mathrm{sf}_G\{D(\mathcal{F}_X)+\mathcal{A}_X, D(\mathcal{F}_X)+T_0\mathcal{V}+\mathcal{A}_{T_0,Y}\}).\end{aligned} \tag{4.29}$$

Since $D(\mathcal{F}_X)+T\mathcal{V}+\mathcal{A}_{T,Y}$ is invertible for $T \geq T_0$, the proof of (4.25) is the same as [24, Theorem 2.22]. Indeed, as in [25, (6.8)], for $u' > 0$ fixed, there exist $C > 0$, $T' \geq T_0$ and $\delta > 0$ such that for $u \geq u'$ and $T \geq T'$, we have

$$|\beta_g^T(T,u)| \leq \frac{C}{T^{1+\delta}}\exp(-cu^2). \tag{4.30}$$

The proof of Theorem 4.4 is completed. □

Theorem 4.5 (i) *For any $u \in (0,1]$, there exist $C > 0$ and $\delta > 0$ such that, for T large enough, we have*

$$|\beta_g^T(T,u)| \leq \frac{C}{T^{1+\delta}}. \tag{4.31}$$

(ii) *There exist $C > 0$, $\gamma \in (0,1]$ such that for $u \in (0,1]$, $0 \leq T \leq u^{-1}$,*

$$\begin{aligned}&\left|u^{-1}\beta_g^T\left(\frac{T}{u},u\right)+\frac{1}{2\sqrt{\pi}\sqrt{-1}}\cdot\frac{2^{\ell_2/2}}{\det^{1/2}(1-g|_{N_{X^g/X}})}\int_{X^g}\widehat{\mathrm{A}}_g(TX,\nabla^{TX})\right.\\ &\left.\cdot\,\psi_{X^g}\mathrm{Tr}^{\mathcal{E}_X/\mathcal{S}}\left[\sigma_{\ell_2}(g^{\mathcal{E}_X})\mathcal{V}|_{X^g}\exp\left(-R_{T^2}^{\mathcal{E}_X/\mathcal{S}}|_{X^g}\right)\right]\right| \leq \frac{C(u(1+T))^{\gamma}}{\sup\{T,1\}}.\end{aligned} \tag{4.32}$$

(iii) *For any* $T > 0$,

$$\lim_{u\to 0} u^{-2}\beta_g^T(T/u^2, u) = 0. \tag{4.33}$$

(iv) *There exist* $C > 0$, $\delta \in (0, 1]$ *such that for* $u \in (0, 1]$, $T \geq 1$,

$$\left|u^{-2}\beta_g^T(T/u^2, u)\right| \leq \frac{C}{T^{1+\delta}}. \tag{4.34}$$

Proof It is easy to see that (i) follows directly from (4.30).

Note that in this theorem, $u \in (0, 1]$. By (4.5), the perturbation operator does not appear. So the proof of (ii)–(iv) here are totally the same as that of [13, Theorem 3.10–3.12] except for replacing the reference of [11] there by the corresponding reference of [7].

Remark that the setting of this paper uses the language of Clifford modules, not the spin case in the references. However, there is no additional difficulty for this differences. The reason is that in each proof of Theorem 4.5(ii)–(iv), we localize the problem first. Locally, all manifolds are spin and the Clifford module could be written as (2.20).

The proof of Theorem 4.5 is completed. □

Now we use the estimates in Theorems 4.3–4.5 to prove Theorem 4.1.

Proof of Theorem 4.1 From (4.8), we know that

$$\int_\varepsilon^A \beta_g^u(T_1, u)du - \int_0^{T_1} \beta_g^T(T, A)dT - \int_\varepsilon^A \beta_g^u(0, u)du + \int_0^{T_1} \beta_g^T(T, \varepsilon)dT$$
$$= I_1^0 + I_2^0 + I_3^0 + I_4^0 = 0. \tag{4.35}$$

We take the limits $A \to +\infty$, $T_1 \to +\infty$ and then $\varepsilon \to 0$ in the indicated order. Let I_j^k, $j = 1, 2, 3, 4$, $k = 1, 2, 3$ denote the value of the part I_j^0 after the kth limit.

From Theorem 4.3, (4.10) and the dominated convergence theorem, we conclude that

$$I_1^3 = -\tilde{\eta}_g(\mathcal{F}_Y, \mathcal{A}_Y). \tag{4.36}$$

Furthermore, by Theorem 4.4, we get

$$I_2^3 = -\operatorname{ch}_g(\operatorname{sf}_G\{D(\mathcal{F}_X) + \mathcal{A}_X, D(\mathcal{F}_X) + T_0\mathcal{V} + \mathcal{A}_{T_0,Y}\}). \tag{4.37}$$

From (4.6), we obtain that

$$I_3^3 = \tilde{\eta}_g(\mathcal{F}_X, \mathcal{A}_X). \tag{4.38}$$

Finally, we calculate the last part. By definition,

$$I_4^1 = I_4^0 = \int_0^{T_1} \beta_g^T(T, \varepsilon)dT. \tag{4.39}$$

As $T_1 \to +\infty$, by Theorem 4.5(i),

$$I_4^2 = \int_0^{+\infty} \beta_g^T(T, \varepsilon)dT = \int_0^{+\infty} \varepsilon^{-1}\beta_g^T(T/\varepsilon, \varepsilon)dT. \tag{4.40}$$

Set

$$K_1 = \int_0^1 \varepsilon^{-1} \beta_g^T(T/\varepsilon, \varepsilon) dT,$$
$$K_2 = \int_\varepsilon^1 \varepsilon^{-2} \beta_g^T(T/\varepsilon^2, \varepsilon) dT, \tag{4.41}$$
$$K_3 = \int_1^{+\infty} \varepsilon^{-2} \beta_g^T(T/\varepsilon^2, \varepsilon) dT.$$

Clearly,

$$I_4^2 = K_1 + K_2 + K_3. \tag{4.42}$$

To simplify the notation, we denote by

$$\mathcal{D}(T) := \frac{1}{2\sqrt{\pi}\sqrt{-1}} \frac{2^{\ell_2/2}}{\det^{1/2}(1 - g|_{N_{X^g/X}})} \cdot \psi_{X^g} \operatorname{Tr}^{\mathcal{E}_X/\mathcal{S}} \left[\sigma_{\ell_2}(g^{\mathcal{E}_X}) \mathcal{V}|_{X^g} \exp\left(-R_{T^2}^{\mathcal{E}_X/\mathcal{S}}|_{X^g} \right) \right]. \tag{4.43}$$

Then by Definition 3.3, after changing the variable, we have

$$\gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X) = \int_0^\infty \mathcal{D}(T) dT. \tag{4.44}$$

As $\varepsilon \to 0$, by Theorem 4.5(ii),

$$K_1 \to -\int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX}) \cdot \int_0^1 \mathcal{D}(T) dT. \tag{4.45}$$

We write K_2 in the form

$$K_2 = \int_\varepsilon^1 \frac{T}{\varepsilon} \left\{ \varepsilon^{-1} \beta_g^T(T/\varepsilon^2, \varepsilon) + \int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX}) \mathcal{D}(T/\varepsilon) \right\} \frac{dT}{T} - \int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX}) \int_1^{\varepsilon^{-1}} \mathcal{D}(T) dT. \tag{4.46}$$

By Theorem 4.5(ii), there exist $C > 0$, $\gamma \in (0, 1]$ such that for $0 < \varepsilon \le T \le 1$

$$\left| \frac{T}{\varepsilon} \left\{ \varepsilon^{-1} \beta_g^T(T/\varepsilon^2, \varepsilon) + \int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX}) \mathcal{D}(T/\varepsilon) \right\} \right| \le C \left(\varepsilon \left(1 + \frac{T}{\varepsilon} \right) \right)^\gamma \le C(2T)^\gamma. \tag{4.47}$$

Using Theorem 4.5(iii), (3.25), (4.47) and the dominated convergence theorem, as $\varepsilon \to 0$,

$$K_2 \to -\int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX}) \int_1^{+\infty} \mathcal{D}(T) dT. \tag{4.48}$$

Using Theorem 4.5(iii), (iv) and the dominated convergence theorem, we see that as $\varepsilon \to 0$,

$$K_3 \to 0. \tag{4.49}$$

Combining Definition 3.3, (4.42), (4.45), (4.48) and (4.49), we see that as $\varepsilon \to 0$,

$$I_4^3 = -\int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX})\, \gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X). \tag{4.50}$$

Thus (4.1) follows from (4.35)–(4.38) and (4.50).

The proof of Theorem 4.1 is completed. □

4.2 Proof of Theorem 3.7

In this subsection, we use the functoriality of the equivariant eta forms Theorem 2.7 to reduce the proof of Theorem 3.7 to the case when the base manifold is a point.[1] Recall that we may assume that B is compact.

Lemma 4.6 *There exist a $\mathbb{Z}_2$-graded self-adjoint $C(TB)$-module $(\mathcal{E}_B, h^{\mathcal{E}_B})$ and a positive integer $q \in \mathbb{Z}_+$ such that*

$$\widehat{\mathrm{A}}(TB, \nabla^{TB})\, \mathrm{ch}(\mathcal{E}_B/\mathcal{S}, \nabla^{\mathcal{E}_B}) - q$$

is an exact form for any Euclidean connection ∇^{TB} and Clifford connection $\nabla^{\mathcal{E}_B}$.

Proof Let $(\mathcal{E}_0, h^{\mathcal{E}_0})$ be a $\mathbb{Z}_2$-graded self-adjoint $C(TB)$-module. Let $\nabla^{\mathcal{E}_0}$ be a Clifford connection on $(\mathcal{E}_0, h^{\mathcal{E}_0})$. Then since the G-action is trivial on B, from the definition of the $\widehat{\mathrm{A}}$-genus and (2.33), there exists $m \in \mathbb{Z}$ such that

$$\widehat{\mathrm{A}}(TB, \nabla^{TB}) \cdot \mathrm{ch}(\mathcal{E}_0/\mathcal{S}, \nabla^{\mathcal{E}_0}) = m + \alpha, \tag{4.51}$$

where $\alpha \in \Omega^{\mathrm{even}}(B)$ is a closed form and $\deg \alpha \geq 2$. We choose $\mathcal{E}_0$ such that $m > 0$.[2]

Since α is nilpotent,

$$\{\widehat{\mathrm{A}}(TB, \nabla^{TB})\, \mathrm{ch}(\mathcal{E}_0/\mathcal{S}, \nabla^{\mathcal{E}_0})\}^{-1} = \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^k}{m^{k+1}} \tag{4.52}$$

is a closed well-defined even differential form over B. From the isomorphism

$$\mathrm{ch}\colon K^0(B) \otimes \mathbb{R} \xrightarrow{\simeq} H^{\mathrm{even}}(B, \mathbb{R}), \tag{4.53}$$

there exist positive real number $q \in \mathbb{R}$ and virtual complex vector bundle $E = E_+ - E_-$, such that $q^{-1}\,\mathrm{ch}([E]) = [\{\widehat{\mathrm{A}}(TB, \nabla^{TB})\, \mathrm{ch}(\mathcal{E}_0/\mathcal{S}, \nabla^{\mathcal{E}_0})\}^{-1}]$. Let ∇^E be a connection on E. Let $\mathcal{E}_B = \mathcal{E}_0 \widehat{\otimes} E$ and $\nabla^{\mathcal{E}_B} = \nabla^{\mathcal{E}_0} \otimes 1 + 1 \otimes \nabla^E$. Then $\mathrm{ch}(\mathcal{E}_B/\mathcal{S}, \nabla^{\mathcal{E}_B}) = \mathrm{ch}(\mathcal{E}_0/\mathcal{S}, \nabla^{\mathcal{E}_0})\, \mathrm{ch}(E, \nabla^E)$. So we have

$$[\widehat{\mathrm{A}}(TB, \nabla^{TB})\, \mathrm{ch}(\mathcal{E}_B/\mathcal{S}, \nabla^{\mathcal{E}_B})] = q \in H^{\mathrm{even}}(B, \mathbb{R}).$$

From (4.51), we have $q \in \mathbb{Z}_+$.

The proof of Lemma 4.6 is completed. □

Let $(\mathcal{E}_B, h^{\mathcal{E}_B})$ be the $C(TB)$-module taken in Lemma 4.6. Let $\nabla^{\mathcal{E}_B}$ be a Clifford connection on $(\mathcal{E}_B, h^{\mathcal{E}_B})$. Thus

$$\mathcal{F}_V = (V, \pi_V^* \mathcal{E}_B \widehat{\otimes} \mathcal{E}_X, \pi_V^* g^{TB} \oplus g^{TX}, \pi_V^* h^{\mathcal{E}_B} \otimes h^{\mathcal{E}_X}, \pi_V^* \nabla^{\mathcal{E}_B} \otimes 1 + 1 \otimes \nabla^{\mathcal{E}_X}) \tag{4.54}$$

[1] The author thanks Prof. Xiaonan Ma for pointing out this simplification, which is related to a remark in [8, Section 7.5].

[2] One example is the exterior bundle with the $\mathbb{Z}_2$-grading induced by the Hodge star operator (see e.g., [5, pp. 150]).

is an equivariant geometric family over a point. Let

$$\mathcal{F}_{V,t} = (V, \pi_V^*\mathcal{E}_B \widehat{\otimes} \mathcal{E}_X, g_t^{TV}, \pi_V^* h^{\mathcal{E}_B} \otimes h^{\mathcal{E}_X}, \nabla_t^{\mathcal{E}_V}) \tag{4.55}$$

be the rescaled equivariant geometric family over a point constructed in the same way as in (2.60).

Lemma 4.7 *There exist $t_0 > 0$ and $T' \geq 1$, such that for any $t \geq t_0$ and $T \geq T'$, the operator $D(\mathcal{F}_{V,t}) + tT\mathcal{V} + t\mathcal{A}_{T,Y}$ is invertible.*

Proof Let $f_1, \ldots, f_l$ be a locally orthonormal basis of TB. Let f_p^H be the horizontal lift of f_p on T^HV. Let $e_1, \ldots, e_n$ be a locally orthonormal basis of TX. Set

$$D_t^B = c(f_p)\nabla_{f_p}^{\mathscr{E}_X,u} + \frac{1}{8t}\langle [f_p^H, f_q^H], e_i\rangle c(e_i)c(f_p)c(f_q). \tag{4.56}$$

By [25, (5.6)], we have

$$D(\mathcal{F}_{V,t}) + tT\mathcal{V} + t\mathcal{A}_{T,Y} = t(D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y}) + D_t^B. \tag{4.57}$$

From Proposition 3.6, since B is compact, there exist $c > 0$ and $T' > 0$, such that for any $s \in \Lambda(T^*B)\widehat{\otimes}\mathscr{E}_X$, $T \geq T'$,

$$\|(D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y})s\|_0^2 \geq c^2\|s\|_0^2. \tag{4.58}$$

Let

$$R_{t,T} := t[D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y}, D_t^B] + D_t^{B,2}. \tag{4.59}$$

We have

$$(D(\mathcal{F}_{V,t}) + tT\mathcal{V} + t\mathcal{A}_{T,Y})^2 = t^2(D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y})^2 + R_{t,T}. \tag{4.60}$$

Let $|\cdot|_{T,1}$ be the norm defined in the same way as [8, Definition 9.13]. In particular,

$$\|s\|_0 \leq |s|_{T,1}. \tag{4.61}$$

Note that the perturbation operator $\mathcal{A}_{T,Y}$ is uniformly bounded with respect to $T \geq 1$. From the arguments in the proof of [8, Theorem 9.14], we could obtain that there exist $C_1, C_2, C_3 > 0$, such that for $T \geq 1$, $t \geq 1$, $s \in \Lambda(T^*B)\widehat{\otimes}\mathscr{E}_X$,

$$\begin{aligned} &\|(D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y})s\|_0^2 \geq C_1|s|_{T,1}^2 - C_2\|s\|_0^2, \\ &|\langle R_{t,T}s, s\rangle_0| \leq C_3\, t\, \|s\|_0 \cdot |s|_{T,1}. \end{aligned} \tag{4.62}$$

Take $\alpha = c^2/(c^2 + 2C_2)$. By (4.58)–(4.62), for $T \geq T'$, $t \geq 1$, we have

$$\begin{aligned} &\|(D(\mathcal{F}_{V,t}) + tT\mathcal{V} + t\mathcal{A}_{T,Y})s\|_0^2 = |\langle t^2(D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y})^2 s + R_{t,T}s, s\rangle_0| \\ &\geq (1-\alpha)t^2\|(D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y})s\|_0^2 \\ &\quad + \alpha t^2\|(D(\mathcal{F}_X) + T\mathcal{V} + \mathcal{A}_{T,Y})s\|_0^2 - |\langle R_{t,T}s, s\rangle_0| \\ &\geq (1-\alpha)c^2t^2\|s\|_0^2 + \alpha C_1 t^2|s|_{T,1}^2 - \alpha C_2 t^2\|s\|_0^2 - C_3 t\|s\|_0 \cdot |s|_{T,1} \\ &\geq \alpha C_2 t^2\|s\|_0^2 + t(\alpha C_1 t - C_3)|s|_{T,1}^2. \end{aligned} \tag{4.63}$$

Take $t_0 = \max\{2C_3/\alpha C_1, 1\}$. For any $t \geq t_0$, $T \geq T'$, there exists $C > 0$, such that

$$\|(D(\mathcal{F}_{V,t}) + tT\mathcal{V} + t\mathcal{A}_{T,Y})s\|_0^2 \geq Ct^2\|s\|_0^2. \tag{4.64}$$

Since $D(\mathcal{F}_{V,t}) + tT\mathcal{V} + t\mathcal{A}_{T,Y}$ is self-adjoint, by (4.64), it is surjective. Thus $D(\mathcal{F}_{V,t}) + tT\mathcal{V} + t\mathcal{A}_{T,Y}$ is invertible.

The proof of Lemma 4.7 is completed. □

Let

$$\mathcal{F}_W = (W, \pi_W^*\mathcal{E}_B\widehat{\otimes}\mathcal{E}_Y, \pi_W^* g^{TB} \oplus g^{TY}, \pi_W^* h^{\mathcal{E}_B} \otimes h^{\mathcal{E}_Y}, \pi_W^*\nabla^{\mathcal{E}_B} \otimes 1 + 1 \otimes \nabla^{\mathcal{E}_Y}) \tag{4.65}$$

be the equivariant geometric family over a point. Let

$$\mathcal{F}_{W,t} = (W, \pi_W^*\mathcal{E}_B\widehat{\otimes}\mathcal{E}_Y, g_t^{TW}, \pi_V^* h^{\mathcal{E}_B} \otimes h^{\mathcal{E}_Y}, \nabla_t^{\mathcal{E}_W}) \tag{4.66}$$

be the rescaled equivariant geometric family constructed in the same way as in (2.60) and (4.55).

Let t_0 be the constant taking in Lemma 4.7. We may assume that when $t \geq t_0$, $D(\mathcal{F}_{W,t}) + 1\widehat{\otimes}t\mathcal{A}_Y$ is invertible by the arguments before Theorem 2.7. By Theorem 2.7 and Lemma 4.7, we have

$$\begin{aligned}\widetilde{\eta}_g(\mathcal{F}_{W,t_0}, 1\widehat{\otimes}t_0\mathcal{A}_Y) &= \int_B \widehat{\mathrm{A}}(TB, \nabla^{TB})\, \mathrm{ch}(\mathcal{E}_B/\mathcal{S}, \nabla^{\mathcal{E}_B})\, \widetilde{\eta}_g(\mathcal{F}_Y, \mathcal{A}_Y) \\ &\quad - \int_{W^g} \left(\widetilde{\widehat{\mathrm{A}}}_g \cdot \widetilde{\mathrm{ch}}_g\right)\left(\nabla_{t_0}^{TW}, \nabla^{TB,TY}, \nabla_{t_0}^{\mathcal{E}_W}, {}^0\nabla^{\mathcal{E}_W}\right)\end{aligned} \tag{4.67}$$

and

$$\begin{aligned}&\widetilde{\eta}_g(\mathcal{F}_{V,t_0}, 1\widehat{\otimes}(t_0T'\mathcal{V} + t_0\mathcal{A}_{T',Y})) \\ &= \int_B \widehat{\mathrm{A}}(TB, \nabla^{TB})\, \mathrm{ch}(\mathcal{E}_B/\mathcal{S}, \nabla^{\mathcal{E}_B})\, \widetilde{\eta}_g(\mathcal{F}_X, T'\mathcal{V} + \mathcal{A}_{T',Y}) \\ &\quad - \int_{V^g} \left(\widetilde{\widehat{\mathrm{A}}}_g \cdot \widetilde{\mathrm{ch}}_g\right)\left(\nabla_{t_0}^{TV}, \nabla^{TB,TX}, \nabla_{t_0}^{\mathcal{E}_V}, {}^0\nabla^{\mathcal{E}_V}\right).\end{aligned} \tag{4.68}$$

Set

$$\begin{aligned}\Delta_B &= \tilde{\eta}_g(\mathcal{F}_X, T'\mathcal{V} + A_{T',Y}) - \tilde{\eta}_g(\mathcal{F}_Y, \mathcal{A}_Y) \\ &\quad - \int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX})\, \gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X) \in \Omega^*(B, \mathbb{C})/\mathrm{Im}d.\end{aligned} \tag{4.69}$$

From [5, (1.17)], Theorem 3.4 and (2.45), we have $d^B\Delta_B = 0$.

Recall that $\mathcal{V} \in \mathrm{End}_{C(TX)}(\mathcal{E}_X)$ satisfies the fundamental assumption (3.13) with respect to $\mathcal{F}_Y$ and $\mathcal{F}_X$. Let $1\widehat{\otimes}\mathcal{V}$ is the extension of $\mathcal{V}$ on $\pi_V^*\mathcal{E}_B\widehat{\otimes}\mathcal{E}_X$. Then $1\widehat{\otimes}\mathcal{V}$ satisfies the fundamental assumption (3.13) with respect to $\mathcal{F}_W$ and $\mathcal{F}_V$. Furthermore, $1\widehat{\otimes}t\mathcal{V}$ satisfies the fundamental assumption (3.13) with respect to $\mathcal{F}_{W,t}$ and $\mathcal{F}_{V,t}$. Observe that $\gamma_g^V(\mathcal{F}_{W,t}, \mathcal{F}_{V,t})$ does not depend on t. We also denote it by $\gamma_g^V(\mathcal{F}_W, \mathcal{F}_V)$.

From Theorem 4.1, if dim V is odd, there exists $T_0 > 0$ such that

$$\begin{aligned}\tilde{\eta}_g(\mathcal{F}_{V,t_0}, \mathcal{A}_V) &= \tilde{\eta}_g(\mathcal{F}_{W,t_0}, 1\widehat{\otimes}t_0\mathcal{A}_Y) + \int_{V^g} \widehat{\mathrm{A}}_g(TX, \nabla_{t_0}^{TX})\, \gamma_g^X(\mathcal{F}_W, \mathcal{F}_V) \\ &\quad + \mathrm{ch}_g(\mathrm{sf}_G\{D(\mathcal{F}_{V,t_0}) + \mathcal{A}_V, D(\mathcal{F}_{V,t_0}) + 1\widehat{\otimes}(T_0t_0\mathcal{V} + t_0\mathcal{A}_{T_0,Y})\}).\end{aligned} \tag{4.70}$$

We may assume that $T_0 \geq T'$, which is determined in Lemma 4.7. By anomaly formula Theorem 2.6, we have

$$\begin{aligned}\tilde{\eta}_g(\mathcal{F}_{V,t_0}, 1\widehat{\otimes}(T_0 t_0\mathcal{V} + t_0\mathcal{A}_{T_0,Y})) &= \tilde{\eta}_g(\mathcal{F}_{W,t_0}, 1\widehat{\otimes} t_0\mathcal{A}_Y) \\ &\quad + \int_{V^g} \widehat{\mathrm{A}}_g(TV, \nabla_{t_0}^{TV})\, \gamma_g^V(\mathcal{F}_W, \mathcal{F}_V). \end{aligned} \tag{4.71}$$

Note that if dim V is even, (4.71) also holds, because in this case all terms in (4.71) vanish.

From the anomaly formula Theorem 2.6, Lemma 4.7 and (4.71), for $t > t_0$, we have

$$\begin{aligned}&\int_{V^g} \left(\widetilde{\widehat{\mathrm{A}}}_g \cdot \widetilde{\mathrm{ch}}_g\right)\left(\nabla_{t_0}^{TV}, \nabla_t^{TV}, \nabla_{t_0}^{\mathcal{E}_V}, \nabla_t^{\mathcal{E}_V}\right) \\ &= \int_{W^g} \left(\widetilde{\widehat{\mathrm{A}}}_g \cdot \widetilde{\mathrm{ch}}_g\right)\left(\nabla_{t_0}^{TW}, \nabla_t^{TW}, \nabla_{t_0}^{\mathcal{E}_W}, \nabla_t^{\mathcal{E}_W}\right) \\ &\quad - \int_{V^g} \widehat{\mathrm{A}}_g(TV, \nabla_{t_0}^{TV})\, \gamma_g^V(\mathcal{F}_W, \mathcal{F}_V) + \int_{V^g} \widehat{\mathrm{A}}_g(TV, \nabla_t^{TV})\, \gamma_g^V(\mathcal{F}_W, \mathcal{F}_V). \end{aligned} \tag{4.72}$$

Note that locally the manifolds are spin. From [25, Proposition 4.5] and the arguments in [25, Section 5.5], we have

$$\begin{aligned}&\lim_{t\to+\infty} \left(\widetilde{\widehat{\mathrm{A}}}_g \cdot \widetilde{\mathrm{ch}}_g\right)\left(\nabla_{t_0}^{TV}, \nabla_t^{TV}, \nabla_{t_0}^{\mathcal{E}_V}, \nabla_t^{\mathcal{E}_V}\right) \\ &= \left(\widetilde{\widehat{\mathrm{A}}}_g \cdot \widetilde{\mathrm{ch}}_g\right)\left(\nabla_{t_0}^{TV}, \nabla^{TB,TX}, \nabla_{t_0}^{\mathcal{E}_V}, {}^0\nabla^{\mathcal{E}_V}\right)\end{aligned} \tag{4.73}$$

and

$$\begin{aligned}\lim_{t\to+\infty} \widehat{\mathrm{A}}_g(TV, \nabla_t^{TV}) &= \widehat{\mathrm{A}}_g(TV, \nabla^{TB,TX}) \\ &= \pi_V^* \widehat{\mathrm{A}}(TB, \nabla^{TB}) \cdot \widehat{\mathrm{A}}_g(TX, \nabla^{TX}). \end{aligned} \tag{4.74}$$

By Definition 3.3, we have

$$\gamma_g^V(\mathcal{F}_W, \mathcal{F}_V) = \mathrm{ch}(\mathcal{E}_B/\mathcal{S}, \nabla^{\mathcal{E}_B})\gamma_g^X(\mathcal{F}_Y, \mathcal{F}_X). \tag{4.75}$$

From Lemma 4.6 and (4.67)–(4.75), since B is compact, we have

$$\int_B \Delta_B = q^{-1} \cdot \int_B \widehat{\mathrm{A}}(TB, \nabla^{TB})\, \mathrm{ch}(\mathcal{E}_B/\mathcal{S}, \nabla^{\mathcal{E}_B}) \cdot \Delta_B = 0. \tag{4.76}$$

Let K be a compact oriented submanifold of B. Let $\mathcal{F}_Y|_K$ and $\mathcal{F}_X|_K$ be the restrictions of $\mathcal{F}_Y$ and $\mathcal{F}_X$ on K. Let $T_0 \geq 1$ be the constant determined in Proposition 3.6 associated with B. Then $(T\mathcal{V} + \mathcal{A}_{T,Y})|_K$ is the perturbation operator with respect to $D(\mathcal{F}_X|_K)$ over K for $T \geq T_0$. Set

$$\begin{aligned}\Delta_K &= \tilde{\eta}_g(\mathcal{F}_X|_K, (T_0\mathcal{V} + A_{T_0,Y})|_K) - \tilde{\eta}_g(\mathcal{F}_Y|_K, \mathcal{A}_Y|_K) \\ &\quad + \int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX})\, \gamma_g^X(\mathcal{F}_Y|_K, \mathcal{F}_X|_K) \in \Omega^*(K, \mathbb{C})/\mathrm{Im}d. \end{aligned} \tag{4.77}$$

From Definition 2.4 and 3.3, we could see that $\int_K \Delta_B = \int_K \Delta_K$.

On the other hand, from (4.76), we have $\int_K \Delta_K = 0$. So for any compact oriented submanifold K of B, we have

$$\int_K \Delta_B = 0. \tag{4.78}$$

By a result of Thom [30, Theorem 2.29], for any homology class $h \in H_*(B, \mathbb{Z})$, there is $n \in \mathbb{Z}$ and a compact oriented submanifold K such that K presents nh. Thus Δ_B is exact on B. Therefore, we obtain Theorem 3.7 from the anomaly formula Theorem 2.6.

The proof of our main result is completed.

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